On Intuitionistic Fuzzy G-Modules On GF($p^n$)

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ABSTRACT. In this paper, we have constructed an intuitionistic fuzzy G-module with level cardinality ($n+1$) on the Galois field GF($p^n$), and then proved that infinite many such intuitionistic fuzzy G-modules can be constructed on it. We have also proved that each such intuitionistic fuzzy G-module, admits a sequence of $k$ intuitionistic fuzzy G-submodules, where $k$ is the number of divisors of $n$. Further, we have also discussed intuitionistic fuzzy noetherian G-module on GF($p^n$).

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1. Introduction

It is well-known result that there exists finite field of order $q$ if and only if $q$ is of the form $p^n$, where $p$ is a prime number and $n$ is a positive integer. Such a field is called Galois field and is denoted by GF($p^n$). The notion of intuitionistic fuzzy G-modules and their properties are discussed by the author et.al. in [4, 5, 6, 7, 8]. In this paper, we construct an intuitionistic fuzzy G-module of level cardinality ($n+1$). We also proved that each such intuitionistic fuzzy G-module, admits a sequence of $k$ intuitionistic fuzzy G-submodules where $k$ is the number of divisors of $n$. Further, we have also discussed intuitionistic fuzzy noetherian G-module on GF($p^n$).

2. Preliminaries

In this section, we first discuss some important results and properties of Galois field GF($p^n$), G-modules, intuitionistic fuzzy set theory and intuitionistic fuzzy G-modules, which are respectively taken from [9], [3], [1, 2], [4, 5, 6].

Definition 2.1. ([9]) A field $K$ with $p^n$ elements is called a Galois field and is denoted by GF($p^n$), where $p$ being a positive prime number.

Theorem 2.2. ([9]) Let $p$ be a prime number and $n$ be a positive integer. Then there exists a field with $p^n$ elements.

Theorem 2.3. ([9]) The multiplicative group of Galois field is cyclic.

Theorem 2.4. ([9]) Let $K'$ be a subfield of the Galois field GF($p^n$). Then there exists an integer $m$ such that $K'$ contains $p^m$ elements and $m$ divides $n$. 
Remark 2.5. ([9]) Any finite field having \( p^n \) elements \((p \text{ is prime})\) has a subfield isomorphic to \( Z_p \).

Definition 2.6. ([3]) Let \( G \) be a group and \( M \) be a vector space over a field \( K \). Then \( M \) is called a \( G \)-module if for every \( g \in G \) and \( m \in M \), \( \exists \) a product (called the action of \( G \) on \( M \)), \( gm \in M \) satisfies the following axioms

(i): \( 1_G \cdot m = m \), \( \forall \ m \in M \) ( \( 1_G \) being the identity of \( G \))

(ii): \( (g \cdot h) \cdot m = g \cdot (h \cdot m) \), \( \forall \ m \in M, g, h \in G \)

(iii): \( g(k_1m_1 + k_2m_2) = k_1(g.m_1) + k_2(g.m_2) \), \( \forall \ k_1, k_2 \in K; m_1, m_2 \in M \) and \( g \in G \)

Example 2.7. For any prime \( p \), we have \( M = (Z_p, \times_p, +_p) \), is a field. Let \( G = M - \{0\} \). Then under the field operations of \( M \), it is a \( G \)-module.

Example 2.8. For the prime 2, let \( M \) be the field having \( 2^4 = 16 \) elements i.e., \( M = \{ \text{zeros of the polynomial } x^{16} - x \text{ over } Z_2 \} \). Let \( M^* = \{ \text{zeros of the polynomial } x^4 - x \text{ over } Z_2 \} \). Then \( M^* \) is the field having \( 2^4 = 4 \) elements. Hence by theorem (2.4) \( M^* \) is a subfield of \( M \). Let \( G^* = M^* - \{0\} \). Then \( M \) is \( G^* \)-module. Also, \( M \) has a subfield \( K \) isomorphic to \( Z_2 \). If \( G^{**} = K - \{0\} \), then \( M \) is also a \( G^{**} \)-module.

Example 2.9. ([4],[5]) Let \( G = \{1, -1, i, -i\} \) and \( M = C^n (n \geq 1) \). Then \( M \) is a vector space over \( C \), and under the usual addition and multiplication of the elements of \( M \), we can show that \( M \) is a \( G \)-module.

Example 2.10. Consider the Galois field \( M = GF(p^n) \). Then \( M \) is a vector space over \( K = GF(p) \cong Z_p \), the field of integers modulo \( p \). Let \( G = K^* \) the multiplicative group of \( M \). Then we can show that \( M \) is a \( G \)-module.

Let the divisors of \( n \) be \( 1 = d_1, d_2, \ldots, d_k = n \) such that \( 1 = d_1 < d_2 < \ldots < d_k = n \). Let \( G = Z_p - \{0\} \). Then we can show that \( M \) has "k" \( G \)-submodules \( M_i = GF(p^{d_i}) \) for \( i = 1, 2, \ldots, k \).

Definition 2.11. ([3],[9]) Let \( M \) be a \( G \)-module. The \( G \)-submodules of \( M \) are said to satisfy the ascending chain condition (A.C.C) if any chain of \( G \)-submodules of \( M, M_1 \subseteq M_2 \subseteq \ldots \ldots \), terminates. This means that there exists a positive integer \( k \) such that \( M_k = M_n \) for \( k \geq n \). If \( G \)-submodules of \( M \) satisfy the A.C.C then \( M \) is said to be a Noetherian module.

Example 2.12. Every finite dimensional vector space \( V \) over a field \( K \) is Noetherian module. In particular, \( M = GF(p^n) \) as \( G \)-module over \( GF(p) \) is a Noetherian module, where \( G = K^* \) is the multiplicative group of \( M \).

Definition 2.13. ([1],[2]) Let \( X \) be a non-empty set. An intuitionistic fuzzy set (IFS) \( A \) of \( X \) is an object of the form \( A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \), where \( \mu_A : X \rightarrow [0,1] \) and \( \nu_A : X \rightarrow [0,1] \) define the degree of membership and degree of non-membership of the element \( x \in X \) respectively and for any \( x \in X \), we have \( \mu_A(x) + \nu_A(x) \leq 1 \).

Remark 2.14.

(i) When \( \mu_A(x) + \nu_A(x) = 1 \), i.e., \( \nu_A(x) = 1 - \mu_A(x), \forall x \in X \). Then \( A \) is called a fuzzy set.
(ii) For convenience, we write the IFS \( A = \{ x, \mu_A(x), \nu_A(x) : x \in X \} \) by \( A = (\mu_A, \nu_A) \).

**Definition 2.15.** Let \( G \) be a group and \( M \) be a \( G \)-module over \( K \), which is a subfield of \( C \). Then a intuitionistic fuzzy \( G \)-module on \( M \) is an intuitionistic fuzzy set \( A = (\mu_A, \nu_A) \) of \( M \) such that following conditions are satisfied

(i) \( \mu_A(ax + by) \geq \min(\mu_A(x), \mu_A(y)) \) and \( \nu_A(ax + by) \leq \max(\nu_A(x), \nu_A(y)) \), \( \forall a, b \in K \) and \( x, y \in M \)

(ii) \( \mu_A(gm) \geq \mu_A(m) \) and \( \nu_A(gm) \leq \nu_A(m) \), \( \forall g \in G; m \in M \).

**Example 2.16.** ([4]) Let \( G = \{1, -1\}, M = R^n \) over \( R \). Then \( M \) is a \( G \)-module. Define the intuitionistic fuzzy set \( A = (\mu_A, \nu_A) \) on \( M \) by

\[
\mu_A(x) = \begin{cases} 
1, & \text{if } x = 0 \\
0.5, & \text{if } x \neq 0
\end{cases} ; \quad \nu_A(x) = \begin{cases} 
0, & \text{if } x = 0 \\
0.25, & \text{if } x \neq 0
\end{cases}
\]

where \( x = (x_1, x_2, ..., x_n) \in R^n \). Then \( A \) is an intuitionistic fuzzy \( G \)-module on \( M \).

**Theorem 2.17.** ([6]) Consider a maximal chain of submodules of \( G \)-module \( M \) over the field \( K \)

\[ M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M, \]

where \( \subset \) denotes proper inclusion. Then there exists an intuitionistic fuzzy \( G \)-module \( A \) of \( M \) given by

\[
\mu_A(x) = \begin{cases} 
\alpha_0 & \text{if } x \in M_0 \\
\alpha_1 & \text{if } x \in M_1 \setminus M_0 \\
\alpha_2 & \text{if } x \in M_2 \setminus M_1 \\
\ldots & \ldots \ldots \\
\alpha_n & \text{if } x \in M_n \setminus M_{n-1}
\end{cases} ; \quad \nu_A(x) = \begin{cases} 
\beta_0 & \text{if } x \in M_0 \\
\beta_1 & \text{if } x \in M_1 \setminus M_0 \\
\beta_2 & \text{if } x \in M_2 \setminus M_1 \\
\ldots & \ldots \ldots \\
\beta_n & \text{if } x \in M_n \setminus M_{n-1}
\end{cases}
\]

where \( \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \) and \( \beta_0 \leq \beta_1 \leq \beta_2 \leq \beta_n \); \( \alpha_i, \beta_i \in [0, 1] \) such that \( \alpha_i + \beta_i \leq 1, \forall i = 0, 1, ..., n \).

**Remark 2.18.** ([6]) The converse of above theorem (3.5) is also true i.e., any intuitionistic fuzzy \( G \)-module \( A \) of a \( G \)-module \( M \) can be expressed in the above form.

**Definition 2.19.** ([6]) Let \( A \) be an intuitionistic fuzzy set of a \( G \)-module \( M \). Put \( \wedge(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\} \), where \( \alpha_i, \beta_i \in [0, 1] \) such that

\( \alpha_i + \beta_i \leq 1, \forall i = 0, 1, ..., n \) then we call the chain \( (\alpha_0, \beta_0) \geq (\alpha_1, \beta_1) \geq (\alpha_2, \beta_2) \geq \ldots \geq (\alpha_n, \beta_n) \) a double keychain if and only if \( \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \) and \( \beta_0 \leq \beta_1 \leq \beta_2 \leq \beta_n \) and the pair \( (\alpha_i, \beta_i) \) are called double pinned flags for the intuitionistic fuzzy set \( A \). The number \( |\wedge(A)| = n + 1 \) is called the level cardinality of the intuitionistic fuzzy set \( A \).

**Example 2.20.** ([6]) Consider the \( G \)-module \( M = R(i) = C \) over the field \( R \) and let \( G = \{1, -1\} \) be the group. Define an intuitionistic fuzzy set \( A = (\mu_A, \nu_A) \) on \( M \).
defined by
\[ \mu_A(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0.5, & \text{if } z \in R - \{0\} \\ 0.25, & \text{if } z \in R(i) - R \end{cases} \quad \nu_A(z) = \begin{cases} 0, & \text{if } z = 0 \\ 0.25, & \text{if } z \in R - \{0\} \\ 0.5, & \text{if } z \in R(i) - R. \end{cases} \]

Then \( A \) is an intuitionistic fuzzy \( G \)-module on \( M \) of level cardinality \( |\land (A)| = 3 \).

3. Intuitionistic Fuzzy Galois Module

In this section, we construct an intuitionistic fuzzy \( G \)-module \( A \) on Galois field \( GF(p^n) \) and also show that infinite many such intuitionistic fuzzy \( G \)-modules can be constructed. We have also discussed intuitionistic fuzzy noetherian \( G \)-module on \( GF(p^n) \).

Proposition 3.1. Any \( n \)-dimensional \( G \)-module \( M \) over \( K \) has an intuitionistic fuzzy \( G \)-module \( A \) of level cardinality \( |\land (A)| = n + 1 \).

Proof. Let \( \{m_1, m_2, \ldots, m_n\} \) be the basis of \( G \)-module \( M \). Let \( M_i \) be the \( G \)-submodule of \( M \) span by \( \{m_1, m_2, \ldots, m_i\} \). Take \( M_0 = \{0\} \). Then we get a maximal chain of \( G \)-submodules of \( M \) as \( M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M \). Let \( \land (A) = \{(1,0),(1/2,1/n+1),(1/3,1/n),\ldots,(1/n,1,1/2)\} \) be the set of double pinned flags for the intuitionistic fuzzy set \( A = (\mu_A, \nu_A) \) defined by

\[ \mu_A(m) = \begin{cases} 1, & \text{if } m = M_0 = \{0\} \\ 1/2, & \text{if } m = M_1 \setminus M_0 \\ 1/3, & \text{if } m = M_2 \setminus M_1 \\ \vdots \quad \begin{array}{c} \ldots \end{array} \quad \begin{array}{c} \ldots \end{array} \\ 1/n, & \text{if } m = M_{n-1} \setminus M_{n-2} \\ 1/n + 1, & \text{if } m = M_n \setminus M_{n-1} \end{cases} \quad \nu_A(m) = \begin{cases} 0, & \text{if } m = M_0 = \{0\} \\ 1/n + 1, & \text{if } m = M_1 \setminus M_0 \\ 1/n, & \text{if } m = M_2 \setminus M_1 \\ \vdots \quad \begin{array}{c} \ldots \end{array} \quad \begin{array}{c} \ldots \end{array} \\ 1/3, & \text{if } m = M_{n-1} \setminus M_{n-2} \\ 1/2, & \text{if } m = M_n \setminus M_{n-1} \end{cases} \]

i.e., if \( m = c_1m_1 + c_2m_2 + \ldots + c_nm_n \), then

\[ \mu_A(c_1m_1 + c_2m_2 + \ldots + c_nm_n) = \begin{cases} 1, & \text{if } c_i = 0 \forall i \\ 1/2, & \text{if } c_1 \neq 0, c_2 = c_3 = \ldots = c_n = 0 \\ 1/3, & \text{if } c_2 \neq 0, c_3 = c_4 = \ldots = c_n = 0 \\ \vdots \quad \begin{array}{c} \ldots \end{array} \quad \begin{array}{c} \ldots \end{array} \\ 1/n, & \text{if } c_{n-1} \neq 0, c_n = 0 \\ 1/n + 1, & \text{if } c_n \neq 0 \end{cases} \quad \nu_A(c_1m_1 + c_2m_2 + \ldots + c_nm_n) = \begin{cases} 0, & \text{if } c_i = 0 \forall i \\ 1/n + 1, & \text{if } c_1 \neq 0, c_2 = c_3 = \ldots = c_n = 0 \\ 1/n, & \text{if } c_2 \neq 0, c_3 = c_4 = \ldots = c_n = 0 \\ \vdots \quad \begin{array}{c} \ldots \end{array} \quad \begin{array}{c} \ldots \end{array} \\ 1/3, & \text{if } c_{n-1} \neq 0, c_n = 0 \\ 1/2, & \text{if } c_n \neq 0. \end{cases} \]

Then, \( A \) is an intuitionistic fuzzy \( G \)-module of level cardinality \( |\land (A)| = n + 1 \). \( \square \)
Theorem 3.2. For every prime number $p$ and every positive integer $n$, there exists an intuitionistic fuzzy G-module $A$ on $GF(p^n)$ of level cardinality $|\wedge (A)| = n + 1$

Proof. It follows from Proposition (3.1) by taking $K = GF(p^n)$.

Proposition 3.3. For any intuitionistic fuzzy G-module $A$ on a $G$-module $M$ and for each $r \in (0, 1]$, the IFS $A_r = (\mu_A, \nu_A)$ defined by $\mu_A(x) = r\mu_A(x)$ and $\nu_A(x) = (1 - r)\nu_A(x), \forall x \in M$. is also an intuitionistic fuzzy G-module on $M$.

Proof. Let $a, b \in K, x, y \in M$ be any elements, then

$\mu_A(ax + by) = r\mu_A(ax + by) \geq r(\mu_A(x) \land \mu_A(y)) = r\mu_A(x) \land r\mu_A(y) = \mu_A(x) \land \mu_A(y)$

and

$\nu_A(ax + by) = (1 - r)\nu_A(ax + by) \leq (1 - r)(\nu_A(x) \lor \nu_A(y)) = (1 - r)\nu_A(x) \lor (1 - r)\nu_A(y) = \nu_A(x) \lor \nu_A(y)$.

Let $g \in G$ and $x \in M$ be any elements, we have

$\mu_A(gx) = r\mu_A(gx) \geq r\mu_A(x) = \mu_A(x)$ and

$\nu_A(gx) = (1 - r)\nu_A(gx) \leq (1 - r)\nu_A(x) = \nu_A(x)$.

Hence $A_r$ is an intuitionistic fuzzy G-module on $M$.

Remark 3.4. It is easy to check that if in the proposition (3.4), we have $r, s \in (0, 1]$ such that $r < s$ then $A_r \subset A_s$.

Theorem 3.5. For every prime number $p$ and every positive integer $n$, there exists infinite many intuitionistic fuzzy Galois G-module $A_r, r \in (0, 1]$ of level cardinality $|\wedge (A_r)| = n + 1$

Proof. Follows from Theorem (3.2) and Proposition (3.4).

Theorem 3.6. For every prime number $p$ and every positive integer $n$, any intuitionistic fuzzy G-module $A$ on $GF(p^n)$ has a sequence of intuitionistic fuzzy G-submodules $A_j, j = 1, 2, \ldots, k$, where $k$ is the number of divisors of $n$.

Proof. Consider the Galois field $M = GF(p^n)$. Then $M$ is a vector space over $K = GF(p) \cong Z_p$, the field of integers modulo $p$ and $dim_K M = n$. Without loss of generality, we assume that $A$ is an intuitionistic fuzzy G-module in Theorem (3.1). Let the divisors of $n$ be $1 = d_1, d_2, \ldots, d_k = n$ such that $d_1 < d_2 < \ldots < d_k$. Then from theorem (2.13) $M$ has $k$ G-submodules $M_j = GF(p^{d_j})$ for $j = 1, 2, \ldots, k$ such that $Z_p \cong M_1 \subset M_2 \subset \ldots \subset M_k$. Clearly, $M_j$ is a subspace of $M$ of dimension $d_j$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{d_j}\}$ be a basis of $M_j$. Then we can extend this to form a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{d_j}, \ldots, \alpha_n\}$ for $M$. Define an intuitionistic fuzzy set $A_j$ on $M_j$ by

$\mu_{A_j}(c_1\alpha_1 + c_1\alpha_1 + \ldots + c_{d_j}\alpha_{d_j}) = \begin{cases} 1, & \text{if } c_i = 0 \forall i \\ 1/2, & \text{if } c_1 \neq 0, c_2 = c_3 = 0, c_{d_j} = 0 \\ 1/3, & \text{if } c_2 \neq 0, c_3 = c_4 = 0, c_{d_j} = 0 \\ \ldots, \ldots & \text{and} \\ 1/d_j, & \text{if } c_{d_{j-1}} \neq 0, c_{d_j} = 0 \\ 1/d_j + 1, & \text{if } c_{d_j} \neq 0 \end{cases}$
Every intuitionistic fuzzy Galois G-module has an ascending chain of intuitionistic fuzzy G-submodules, which terminates.

Proof. By theorem (3.2), for every prime number \( p \) and every positive integer \( n \), there exists an intuitionistic fuzzy G-module \( A \) on \( GF(p^n) \) of level cardinality \( |\land(A)| = n + 1 \). Also, by theorem (3.7) any intuitionistic fuzzy G-module \( A \) on \( GF(p^n) \) has a sequence of intuitionistic fuzzy G-submodules \( A_j, j = 1, 2, \ldots, k \), where \( k \) is the number of divisors of \( n \).

Let \( t_j = 1/(d_j + 1) \) for \( j = 1, 2, \ldots, k \). Then for each \( j \), we have an IFS \( B_j \) on \( M_j \) defined by

\[
\nu_{B_j}(x) = \begin{cases} 
\mu_{A_j}(x), & \text{if } x \in M_j \\
\frac{1}{t_j}, & \text{if } x \in M - M_j 
\end{cases}
\]

Clearly, each \( B_j \) is an intuitionistic fuzzy G-module on \( M \). Let \( C_j = B_j|_{M_j} \), for \( j = 1, 2, \ldots, k \). Then each \( C_j \) is an intuitionistic fuzzy G-module on \( M_j \) such that

\( C_1 \subseteq C_2 \subseteq \ldots \) terminate at \( k \).

Corollary 3.8. For an intuitionistic fuzzy Galois G-module there exists infinite many chains of intuitionistic fuzzy G-submodules terminates at \( k \).

Proof. Follows from Theorem (3.6) and Theorem (3.8) \( \square \)

4. Conclusions

In this paper, we have constructed an intuitionistic fuzzy G-module of level cardinality \( n+1 \) on the Galois field \( GF(p^n) \), and then proved that infinite many such intuitionistic fuzzy G-modules can be constructed on it. We have also proved that each such an intuitionistic fuzzy G-module, admits a sequence of \( k \) intuitionistic fuzzy G-submodules \( A_j \), where \( k \) is the number of divisors of \( n \). We have also proved that any ascending chain of intuitionistic fuzzy Galois modules terminates at some finite stage and that there are infinitely many such terminating chains of intuitionistic fuzzy G-modules.

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