ON BOUNDED LINEAR OPERATORS IN $b$-HILBERT SPACES AND THEIR NUMERICAL RANGES

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ABSTRACT. In this paper, we introduce the notions of $b$-bounded linear operator, $b$-numerical range and $b$-numerical radius in a $b$-Hilbert space and describe some of their properties. Then we will show that this new numerical range (radius) can be considered as a usual numerical range (radius) in a Hilbert space, so it shares many useful properties with numerical range (radius).

1. INTRODUCTION AND PRELIMINARIES

Quadratic forms and their applications appear in many parts of mathematics and the sciences. A natural extension of these ideas in finite- and infinite-dimensional spaces leads us to the numerical range [7]. The subject has been studied by great mathematicians like K. E. Gustafson, D. K. M. Rao, R. Bahatia, F. Kittaneh, S. S. Dragomir, M. S. Moslehian and others (cf. e.g. [2, 4, 7, 8, 10, 12] and also to the references cited therein), and they have contributed a lot for the extension of this branch of mathematics.

The concept of linear 2-normed spaces was investigated by S. Gähler in 1964 [5], and has been developed extensively in different subjects by many authors [6, 13, 15, 16]. A concept which is closely related to 2-normed space is 2-inner product space which has been intensively studied by many mathematicians in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [3].

In the following we provide some notations, definitions and auxiliary facts which will be used later in this paper.
Definition 1.1. Let $X$ be a linear space of dimension greater than 1 over the field $k$, where $k$ is the real or complex numbers field. Suppose that $\langle \cdot, \cdot | \cdot \rangle$ is a $k$-valued function defined on $X \times X \times X$ satisfying the following conditions:

1. \( \langle x, x | z \rangle \geq 0 \) and \( \langle x, x | z \rangle = 0 \) if and only if $x$ and $z$ are linearly dependent,
2. \( \langle x, y | z \rangle = \langle y, x | z \rangle \),
3. \( \langle x, y | z \rangle = \langle x, y | z \rangle \),
4. \( \langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle \) for all $\alpha \in k$,
5. \( \langle x_1 + x_2, y | z \rangle = \langle x_1, y | z \rangle + \langle x_2, y | z \rangle \).

Then $\langle \cdot, \cdot | \cdot \rangle$ is called a 2-inner product on $X$ and $(X, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space (or 2-pre Hilbert space).

From the definition of 2-inner product it is easy to verify the following assertions:

1. \( \langle 0, y | z \rangle = \langle x, 0 | z \rangle = \langle x, y | 0 \rangle = 0 \).
2. \( \langle x, \alpha y | z \rangle = \overline{\alpha} \langle x, y | z \rangle \).
3. \( \langle x, y | \alpha z \rangle = |\alpha|^2 \langle x, y | z \rangle \), for all $x, y, z \in X$ and $\alpha \in k$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

\[ |\langle x, y | z \rangle|^2 \leq \langle x, x | z \rangle \langle y, y | z \rangle. \]

Example 1.2. (see [1, Example 1.1]) If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, then the standard 2-inner product $\langle \cdot, \cdot | \cdot \rangle$ is defined on $X$ by

\[ \langle x, y | z \rangle = \frac{\langle x, y \rangle \langle x, z \rangle}{\langle z, y \rangle \langle z, z \rangle} = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle, \]

for all $x, y, z \in X$.

In any 2-inner product space $(X, \langle \cdot, \cdot | \cdot \rangle)$ we can define a function $\| \cdot, \cdot \|$ on $X \times X$ by

\[ \|x, z\| = (\langle x, x | z \rangle)^{\frac{1}{2}}, \]

for all $x, z \in X$. It is easy to see that, this function satisfies the following conditions:

1. $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if $x$ and $z$ are linearly dependent,
2. $\|x, z\| = \|z, x\|$,
3. $\|\alpha x, z\| = |\alpha| \|x, z\|$ for all $\alpha \in k$,
4. $\|x_1 + x_2, z\| \leq \|x_1, z\| + \|x_2, z\|$. 

Any function $\| \cdot, \cdot \|$ defined on $X \times X$ and satisfying the conditions (N1)-(N4) is called a
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2-norm on $\mathcal{X}$ and $(\mathcal{X}, \|\cdot\|)$ is called a linear 2-normed space. Whenever a 2-inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is given, we consider it as a linear 2-normed space $(\mathcal{X}, \|\cdot\|)$ with the norm defined by (1.1).

Let $\mathcal{X}$ be a 2-inner product space. A sequence $\{x_n\}$ of $\mathcal{X}$ is said to be convergent if there exists an element $x \in \mathcal{X}$ such that $\lim_{n \to \infty} \|x_n - x, z\| = 0$, for all $z \in \mathcal{X}$. Similarly, we can define a Cauchy sequence in $\mathcal{X}$. A 2-inner product space $\mathcal{X}$ is called a 2-Hilbert space if it is complete. That is, every Cauchy sequence in $\mathcal{X}$ is convergent in this space [13]. Clearly, the limit of any convergent sequence is unique. Now suppose that $b$ is a nonzero fixed vector in $\mathcal{X}$ and take $z = b$, then definition of Cauchy, convergent and 2-Hilbert space change to $b$-Cauchy, $b$-convergent and $b$-Hilbert space [9]. If a sequence $\{x_n\}$ is $b$-convergent to an element of $b$-Hilbert space $\mathcal{X}$ say $x$, then we denote it by $\lim_{n \to \infty} \|b\| x_n = x$.

It is easily verified that in any $b$-Hilbert space $\mathcal{X}$, the mapping $\langle \cdot, \cdot | b \rangle$ is sequentially continuous with respect to semi-norm $\|\cdot, b\|$.

Remark 1.3. (see [1, Pages 127-128]) Assume that $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is a 2-Hilbert space and $L_\xi$ the subspace generated with $\xi$ for a fix element $\xi$ in $\mathcal{X}$. Denote by $\mathcal{M}_\xi$ the algebraic complement of $L_\xi$ in $\mathcal{X}$. So $L_\xi \oplus M_\xi = \mathcal{X}$. We first define the inner product $\langle \cdot, \cdot \rangle_\xi$ on $\mathcal{X}$ as following:

$$\langle x, z \rangle_\xi = \langle x, z | \xi \rangle.$$ 

A straightforward calculations shows that $\langle \cdot, \cdot \rangle_\xi$ is a semi-inner product on $\mathcal{X}$. It is well-known that this semi-inner product induces an inner product on the quotient space $\mathcal{X}/L_\xi$ as

$$\langle x + L_\xi, z + L_\xi \rangle_\xi = \langle x, z \rangle_\xi, \quad (x, z \in \mathcal{X}).$$

By identifying $\mathcal{X}/L_\xi$ with $\mathcal{M}_\xi$ in an obvious way, we obtain an inner product on $\mathcal{M}_\xi$. Define

$$\|x\|_\xi = \sqrt{\langle x, x \rangle_\xi} \quad (x \in \mathcal{M}_\xi).$$

Then $(\mathcal{M}_\xi, \|\cdot\|_\xi)$ is a normed space. Let $\mathcal{X}_\xi$ be the completion of the inner product space $\mathcal{M}_\xi$. For each $b \in \mathcal{X}$, we denote by $L_b$ the subspace generated by $b$. Let $x_1, x_2 \in \mathcal{X}$, then $x_1$ is said to $b$-congruent to $x_2$, if $x_1 - x_2 \in L_b$.

In the present work, we shall introduce the concept of $b$-bounded linear operator and describe some fundamental properties of it. Then we establish $b$-numerical range (radius) for
b-bounded linear operators. This numerical range (radius) can be considered as a usual numerical range (radius) in a Hilbert space, so it shares many useful properties with numerical range (radius).

Throughout this paper, unless otherwise specified, $\mathcal{X}$, $H$ and $L_b^+$ denote $b$-Hilbert space, Hilbert space with the inner product $(.,.)$ chosen to be linear in the first entry, and the orthogonal complement of $L_b$ in $H$, respectively.

2. Main Result

**Definition 2.1.** Let $\mathcal{X}$ be a $b$-Hilbert space. A linear operator $T : \mathcal{X} \to \mathcal{X}$ is called $b$-bounded if $T$ invariants $L_b$ and there is a non-negative real number $M$ such that $\|T(x), b\| \leq M\|x, b\|$ for all $x \in \mathcal{X}$. We define $\|T\|_b$ infimum of such $M$. Obviously,

$$\|T\|_b = \sup\{\|T(x), b\| : \|x, b\| \leq 1\} = \sup\{\|T(x), b\| : \|x, b\| = 1\}.$$ 

We denote the set of all $b$-bounded linear operators on the $b$-Hilbert space $\mathcal{X}$, by $B_b(\mathcal{X})$. It is not hard to see that if $T \in B_b(\mathcal{X})$, then it (sequentially) continuous.

Let $T$ and $T'$ be $b$-bounded linear operators on the $b$-Hilbert space $\mathcal{X}$. They are called equal up to $b$-congruent if $\text{range}(T - T') \subseteq L_b$. Due to the fact $(B_b(\mathcal{X}), \|\|_b)$ is a semi-normed space.

Similarly a linear functional $f : \mathcal{X} \to \mathbb{C}$ is called $b$-bounded if $f(L_b) = \{0\}$ and there is a non-negative real number $M$ such that $|f(x)| \leq M\|x, b\|$ for all $x \in \mathcal{X}$. We define $\|f\|_b$ infimum of such $M$. We observe that $\|f\|_b = \sup\{|f(x)| : \|x, b\| \leq 1\}$ and it defines a norm on the set of all $b$-bounded linear functionals on $\mathcal{X}$ which is denoted by $(\mathcal{X}^*)_b$.

**Example 2.2.** Let $\mathcal{X} = l^2$ together with the standard 2-inner product. Then $\mathcal{X} = l^2$ is a $(1, 0, 0, ...)$-Hilbert space. Assume that $T : \mathcal{X} \to \mathcal{X}$ is a map which is defined by $T(a_1, a_2, ...) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, ...)$. It is readily verified that $T$ is $(1, 0, 0, ...)$-bounded linear operator. Indeed, $\|T((a_1, a_2, ...)), (1, 0, 0, ...)\|^2 = \sum_{n=2}^{\infty} \left(\frac{|a_n|}{n}\right)^2 \leq \sum_{n=2}^{\infty} |a_n|^2 = \|(a_1, a_2, ...), (1, 0, 0, ...)\|^2$.

**Example 2.3.** Let $L^2([-\pi, \pi]) = \{f : [-\pi, \pi] \to \mathbb{R}, \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty\}$ and let $\mathcal{X} = \{f \in L^2([-\pi, \pi]) : f^{(k)} \in L^2([-\pi, \pi]), k = 1, 2, ...\}$. Then $\mathcal{X}$ with the standard 2-inner product is an $e^x$-Hilbert space. Define the operator $T : \mathcal{X} \to \mathcal{X}$ by $T(f) = f'$. An easy computation shows that $T$ invariants $L_{e^x}$ but it is not $e^x$-bounded. Since $\|T(sinx), e^x\|^2 = n^2(\frac{\pi}{2}(e^{2\pi} −$
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\[ e^{-2\pi} - \frac{(e^{x} - e^{-x})^2}{(n + n^3)^2} \] and \[ \|\sin(nx), e^x\|^2 = \frac{\pi}{2}(e^{2\pi} - e^{-2\pi}) - \frac{n^2(e^{x} - e^{-x})^2}{(1 + n^3)^2}, \] then \[ \|T(\sin nx), e^x\| \] goes to infinity as \( n \to \infty. \)

**Proposition 2.4.** Let \( \langle ., . \rangle \) be the standard 2-inner product on the Hilbert space \( H, b \in H \) and \( T \in B(H) \) in which \( T \) reduces \( L_b \), then \( T : (H, \langle ., . \rangle) \to (H, \langle ., . \rangle) \) is a \( b \)-bounded linear operator.

**Proof.** Clearly if range\((T) \subseteq L_b\), then \( \|T\|_b = 0 \). Otherwise, since \( T \in B(H) \), so there is a constant \( M > 0 \) such that \( \|T(x)\| \leq M\|x\| \) for all \( x \in H \). On the other hand, we have \( H = L_b \oplus L_b^\perp \), therefore every element \( x \) of \( H \) can be written uniquely as \( y + z \) for some \( y \in L_b \) and \( z \in L_b^\perp \). Now since \( T \) reduces \( L_b \), then by the definition of standard 2-inner product it follows that

\[
\|(y + z), b\| = \|T(y + z), b\| \leq \|T(y), b\| + \|T(z), b\|
\]

\[
= \|T(z), b\| = (\|T(z), b\|^2)^{\frac{1}{2}} = (\|T(z)\|^2\|b\|^2 - \langle T(z), b \rangle^2)^{\frac{1}{2}}
\]

\[
= \|T(z)\|\|b\| \leq M\|z\|\|b\|.
\]

Cauchy-Schwarz inequality implies that \( |\langle y, b \rangle| = \|y\||\|b\|\), thus we find that

\[
\|x, b\|^2 = \|y + z\|^2\|b\|^2 - |\langle y + z, b \rangle|^2
\]

\[
= (\|y\|^2 + \|z\|^2)\|b\|^2 - |\langle y, b \rangle|^2
\]

\[
= (\|y\|^2 + \|z\|^2)\|b\|^2 - \|y\|^2\|b\|^2 = \|z\|^2\|b\|^2.
\]

By (2.1) and (2.2), we get the desired result. \( \square \)

**Proposition 2.5.** Let \( \langle ., . \rangle \) be the standard 2-inner product on the Hilbert space \( H, b \in H \) and \( T : (H, \langle ., . \rangle) \to (H, \langle ., . \rangle) \) be a \( b \)-bounded linear operator in which invariants \( L_b^\perp \), then \( T \) is a bounded linear operator on \( L_b^\perp \).

**Proof.** First suppose that range\((T) \not\subseteq L_b\). Let \( x \in L_b^\perp \). By virtue of the fact that \( T \) invariants \( L_b^\perp \) and also definition of standard 2-inner product we deduce

\[
\|T(x)\|^2\|b\|^2 = \|T(x)\|^2\|b\|^2 - |\langle T(x), b \rangle|^2 = \|T(x), b\|^2
\]

\[
\leq \|T\|^2\|x, b\|^2 = \|T\|^2\|\|x\|^2\|b\|^2 - |\langle x, b \rangle|^2
\]

\[
= \|T\|^2\|x\|^2\|b\|^2.
\]
Whence \( \|T(x)\| \leq \|T\|_b \|x\| \), for each \( x \in L_b^+ \) and so \( T|_{L_b^+} \) is bounded. Now if range\( (T) \subseteq L_b \), then range\( (T|_{L_b^+}) \subseteq L_b \cap L_b^+ = \{0\} \). It forces that \( T|_{L_b^+} = 0 \). 

Let \( X \) be a \( b \)-Hilbert space. As Remark 1.3, denote by \( M_b \), the algebraic complement of \( L_b \) in \( X \) and identifying \( M_b \) by \( X/L_b \). Also let \( X_b \) be the completion of the inner product space \( M_b \). Let \( T \in B_b(X) \), define the map \( T_b : X_b \to X_b \) by setting \( T_b(z) := \lim_{n \to \infty} T(x_n) + L_b \), where \( z = \lim_{n \to \infty} x_n + L_b \in X_b \). We observe that \( T_b \) is a well-defined linear operator. Clearly 
\[ \|T\|_b = 0, \text{ if range}(T) \subseteq L_b. \]
Otherwise, the inequality
\[ \|(T(x_n) + L_b) - (T(x_m) + L_b)\|_b \leq \|T\|_b \|x_n + L_b\| - \|x_m + L_b\|_b \]
implies that the sequence \( \{T(x_n) + L_b\} \) is Cauchy and so convergent in \( X_b \).

It is rutin to verify that if \( T \) and \( S \) are in \( B_b(X) \) and \( \alpha \) is any scalar in \( \mathbb{k} \), then \( (\alpha T + S)_b = \alpha T_b + S_b \) and \( (TS)_b = T_bS_b \).

According to Remark 1.3, one obtains that \( z = ( \lim_{n \to \infty} \|\cdot\|_b \|x_m\| + L_b, \) where \( z = \lim_{n \to \infty} \|\cdot\|_b \|x_n + L_b\| \in X_b \). By virtue of that fact we get the following result.

**Proposition 2.6.** Let \( X \) be a \( b \)-Hilbert space and \( T \) be a \( b \)-bounded linear operator on \( X \), then \( T_b \) is a bounded linear operator on the Hilbert space \( X_b \) and moreover \( \|T_b\| = \|T\|_b \).

P. K. Harikrishnan et al., [9] proved a version of Riesz representation theorem in framework of \( b \)-Hilbert spaces. By a slightly modification in the proof of [9, Theorem 3.5] we see that this theorem holds for a \( b \)-bounded linear functional defined on a \( b \)-Hilbert space.

**Proposition 2.7.** Let \( X \) be a \( b \)-Hilbert space and \( f \) be a \( b \)-bounded linear functional on \( X \). Then there exists a unique \( y \in X \) up to \( b \)-congruent such that \( f(x) = \langle x,y \rangle \) and \( \|f\|_b = \|y,b\| \).

**Definition 2.8.** Let \( X \) be a \( b \)-Hilbert space. A complex valued function \( B \) on \( X \times X \) is called a conjugate-bilinear functional, if it is linear in the first variable and conjugate-linear in the second. Furthermore, it is called \( b \)-bounded, if \( B(X \times L_b) = B(L_b \times X) = B(L_b \times L_b) = \{0\} \) and there is a nonnegative real number \( M \) such that \( |B(x,y)| \leq M \|x\| \|b\| \|y\| \|b\| \) for all \( x,y \in X \). We denote by \( \|B\|_b \) the infimum of such \( M \). It is easy to verify that \( \|B\|_b = \sup \{|B(x,y)| : x,y \in X, \|x\|_b \leq 1, \|y\|_b \leq 1 \} \). Trivially \( \|\cdot\|_b \) defines a norm on the set of \( b \)-bounded conjugate-bilinear functionals on \( X \). Assume \( S \in B_b(X) \), define \( B_S(x,y) := \langle S(x), y|b\rangle \) for each \( x,y \in X \). It is easy to verify that \( B_S \) is a \( b \)-bounded conjugate-bilinear functional on \( X \) and \( \|B_S\|_b = \|S\|_b \).
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Now we are in a position to investigate existence of an adjoint, which is named $b$-adjoint, for a $b$-bounded linear operator defined on a $b$-Hilbert space. Indeed, we will show that if $\mathcal{X}$ is a $b$-Hilbert space and $T \in B_b(\mathcal{X})$, then there exists a unique $T^* \in B_b(\mathcal{X})$ up to $b$-congruent in which $(T(x), y|b) = \langle x, T^*(y)|b \rangle$ for each $x, y \in \mathcal{X}$. We use a similar method applied in [11, pp. 98-101] for Hilbert spaces in order to obtain a $b$-adjoint for a $b$-bounded linear operator in a $b$-Hilbert space.

Let $\mathcal{X}$ be a $b$-Hilbert space. Consider equivalence relation $\sim$ on $\mathcal{X}$, in which $x \sim y$, if $x, y \in L_b$ and $x \sim x$, if $x \in \mathcal{X} - L_b$. In this case equivalence class $\tilde{\mathcal{X}}$ is $\{L_b, \tilde{x} = \{x\} : x \in \mathcal{X} - L_b\}$. We observe that $(\tilde{\mathcal{X}}, \|\|)$ is a normed space, where

$$\tilde{x} + \tilde{y} = \tilde{x + y}, \quad \tilde{x} + L_b = L_b + \tilde{x} = \tilde{x}, \quad L_b + L_b = L_b,$$

$$\alpha \tilde{x} = \tilde{\alpha x}, \quad \alpha L_b = L_b,$$

$\|L_b\| = 0$ and $\|\tilde{x}\| = \|x, b\|$, for each $x, y \in \mathcal{X} - L_b$ and $\alpha \in k$. Define $\tilde{J} : \tilde{\mathcal{X}} \to (\mathcal{X}^*)_b$ by $\tilde{J}(L_b) = 0$ and if $x \in \mathcal{X} - L_b$, then $\tilde{J}(\tilde{x}) = J_x$, where $J_x(y) = \langle y, x, b \rangle$ for each $y \in \mathcal{X}$. It is easily seen that, $\tilde{J}$ is a surjective isometric conjugate linear operator. Assume that $V : \tilde{\mathcal{X}} \to \mathcal{X}$ defined by $V(L_b) = 0$ and $V(\tilde{x}) = x$ for each $x \in \mathcal{X} - L_b$, clearly $V$ is a linear operator and $\|V\|_b = \sup\{\|V(\tilde{x}), b\| : \|\tilde{x}\| \leq 1\} \leq 1$.

Let $B$ be a $b$-bounded conjugate-bilinear functional on $\mathcal{X}$, $U : \mathcal{X} \to (\mathcal{X}^*)_b$ be defined by $(Ux)(y) := \overline{B(x, y)}$. Then $U$ is a $b$-bounded conjugate linear operator and by Proposition 2.7, for each $x \in \mathcal{X}$, there exists a unique $z \in \mathcal{X}$ up to $b$-congruent in which $Ux = \phi_z$, where $\phi_z(y) = \langle y, z|b \rangle$. Set $S := V\tilde{J}^{-1}U$, it is a $b$-bounded linear operator on $\mathcal{X}$. Indeed we have

$$\|V\tilde{J}^{-1}Ux, b\| \leq \|V\|_b\|\tilde{J}^{-1}Ux\| = \|V\|_b\|Ux\| \leq \sup\{|Ux(y)| : \|y, b\| \leq 1\} < \|B\|_b\|x, b\|,$$

for each $x \in \mathcal{X}$. Now if $B_S(x, y) = \langle S(x), y|b \rangle$, then $B_S$ is a $b$-bounded conjugate-bilinear functional on $\mathcal{X} \times \mathcal{X}$, $\|B_S\|_b = \|S\|_b$ and furthermore, $B_S(x, y) = \langle y, S(x)|b \rangle = \langle y, V\tilde{J}^{-1}Ux|b \rangle = \langle y, z|b \rangle = \overline{Ux(y)} = B(x, y)$. Trivially if $x$ or $y$ are in $L_b$, then $B(x, y) = B_S(x, y) = 0$. Hence every $b$-bounded conjugate bilinear functional is of the form $B_S$ for some $S \in B_b(\mathcal{X})$.

**Theorem 2.9.** Let $T$ be a $b$-bounded linear operator on a $b$-Hilbert space $\mathcal{X}$, then there exists a unique $b$-bounded linear operator $T^* \in B_b(\mathcal{X})$ up to $b$-congruent such that $\langle T(x), y|b \rangle = \langle x, T^*(y)|b \rangle$ for each $x, y \in \mathcal{X}$. In addition, if $S$ and $S'$ are two $b$-adjoints of $T$, then $S_b = S'_b$. 


Proof. Define $B(x, y) = \langle x, T(y) | b \rangle$. It is easily verified that $B$ is a $b$-bounded conjugate-bilinear functional on $X \times X$. So

$$B(x, y) = B_S(x, y) = \langle S(x), y | b \rangle,$$

for some $b$-bounded linear operator $S$ on $X$. Put $T^* := S$, then $T^*$ is a $b$-adjoint of $T$.

Using the same reasoning as [11, Theorem 2.4.1] $b$-adjoint of $T$ is unique up to $b$-congruent. It remains to show that $S_b = S'_b$, for $b$-adjoints $S$ and $S'$ of $T$. For, let $z_1, z_2 \in X_b$, then $z_1 = \lim_{n \to \infty} x_n + L_b$ and $z_2 = \lim_{m \to \infty} y_m + L_b$ for some sequences $\{x_n\}$ and $\{y_m\}$ in $X$. Since $S = S'$ up to $b$-congruent, so for each $n \in \mathbb{N}$, there is a scalar $\mu_n$ in which $S(x_n) = S'(x_n) + \mu_n b$. Thus we have

$$\langle S_b(z_1), z_2 \rangle_b = \langle S_b(\lim_{n \to \infty} x_n + L_b), \lim_{m \to \infty} y_m + L_b) | b \rangle$$

$$= \langle \lim_{n \to \infty} S(x_n) + L_b, \lim_{m \to \infty} y_m + L_b) | b \rangle$$

$$= \lim_{n \to \infty} \lim_{m \to \infty} \langle S'(x_n) + \mu_n b, y_m | b \rangle$$

$$= \lim_{n \to \infty} \lim_{m \to \infty} \langle S'(x_n) + L_b, y_m + L_b) | b \rangle$$

$$= \langle S'_b(z_1), z_2 \rangle_b.$$

It follows that $S_b = S'_b$. \hfill \square

As an immediate consequence of the above theorem we have $T = T^{**}$ up to $b$-congruent.

Let $X$ be a $b$-Hilbert space and $T \in B_b(X)$, then $T$ is called $b$-selfadjoint if $T = T^*$ up to $b$-congruent or equivalently $\langle T(x), y | b \rangle = \langle x, T(y) | b \rangle$ for each $x, y \in X$ and it is called $b$-unitary, if $TT^* = T^*T = I$ (identity operator on $X$) up to $b$-congruent. Note that if $T$ is $b$-unitary, then range$(T) \not\subseteq L_b$.

Now we are ready to establishing $b$-numerical range (radius) for a $b$-bounded linear operator in $b$-Hilbert spaces. To extend a well-known result in Hilbert spaces to $b$-Hilbert spaces.

**Definition 2.10.** Let $T : X \to X$ be a $b$-bounded linear operator on a $b$-Hilbert space $X$. Then $b$-numerical range of $T$ which is denoted by $W_b(T)$ is $\{ \langle T(x), x | b \rangle : x \in X, \| x, b \| = 1 \}$. Also, $b$-numerical radius of $T$ which is denoted by $\omega_b(T)$ is sup$\{ | \langle T(x), x | b \rangle | : x \in X, \| x, b \| = 1 \}$.

A remarkable fact about $b$-numerical range (radius) is its close relation with numerical range (radius) in the usual sense. Indeed, we have $W_b(T) = W(T_b)$ and $\omega_b(T) = \omega(T_b)$.
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By virtue of this fact every question about $b$-numerical range (radius) in a $b$-Hilbert space can be solved as a question about numerical range (radius) in a Hilbert space.

It is easy to verify that $\omega_b(.)$ is a semi-norm on $B_b(\mathcal{X})$. Furthermore, using Proposition 2.6 and [7, Theorem 1.3.1], we have $\omega_b(T) \leq \|T\|_b \leq 2\omega_b(T)$, for each $T \in B_b(\mathcal{X})$.

In the following we extend [7, Theorem 1.2.2] in the framework of $b$-Hilbert spaces.

**Theorem 2.11.** Let $T$ be a $b$-bounded linear operator on a $b$-Hilbert space $\mathcal{X}$. Then $T$ is $b$-selfadjoint if and only if $W_b(T) \subseteq \mathbb{R}$.

**Proof.** Let $z_1 = \lim_{n \to \infty} x_n + L_b$ and $z_2 = \lim_{n \to \infty} y_n + L_b$ be arbitrary elements in $\mathcal{X}_b$. We get

$$
\langle z_1, (T_b)^*(z_2) \rangle_b = \langle T_b (\lim_{n \to \infty} x_n + L_b), \lim_{n \to \infty} y_n + L_b \rangle_b
$$

$$
= \langle \lim_{n \to \infty} T(x_n), \lim_{n \to \infty} y_n | b \rangle
$$

$$
= \langle \lim_{n \to \infty} x_n, T^* (\lim_{n \to \infty} y_n) | b \rangle
$$

$$
= \langle (\lim_{n \to \infty} x_n) + L_b, (\lim_{n \to \infty} T^*(y_n)) + L_b \rangle_b
$$

$$
= \langle \lim_{n \to \infty} x_n + L_b, (T^*)_b (\lim_{n \to \infty} y_n + L_b) \rangle_b
$$

$$
= \langle z_1, (T^*)_b(z_2) \rangle_b.
$$

Therefore $(T_b)^* = (T^*)_b$. Now if $T$ is $b$-selfadjoint, then $(T^*)_b = T_b$ and so $(T_b)^* = T_b$. Applying [7, Theorem 1.2.2] we deduce $W_b(T) = W(T_b) \subseteq \mathbb{R}$. Conversely, if $W_b(T) \subseteq \mathbb{R}$, then $T_b$ is a selfadjoint linear operator on the Hilbert space $\mathcal{X}_b$. That is, $(T_b)^* = T_b$. Consequently for each $x, y \in \mathcal{X}$, $\langle T(x), y | b \rangle = \langle T_b(x + L_b), y + L_b \rangle_b = \langle x + L_b, T_b^*(y + L_b) \rangle_b = \langle x + L_b, T_b(y + L_b) \rangle_b = \langle x, T(y) | b \rangle$. Hence $T$ is $b$-selfadjoint and so the proof is completed. □

In the light of the above discussions we have the following statement.

Suppose that $U$ and $I$ are $b$-unitary and identity operators on a $b$-Hilbert space $\mathcal{X}$, respectively and $T \in B_b(\mathcal{X})$. Then we have

(i) $W_b(\alpha + \beta T) = \alpha + \beta W_b(T)$, for each $\alpha$ and $\beta$ in $\mathbb{K}$.

(ii) $W_b(T^*) = \{ \overline{\mathcal{X}} : \lambda \in W_b(T) \}$.

(iii) $W_b(U^*TU) = W_b(T)$. 

References


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