A New Super Convergent Implicit Runge-Kutta Method for First Order Ordinary Differential Equations

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ABSTRACT--- We present a new efficient super convergent implicit Runge-kutta method (RKM) for solving differential equations (ODEs). Chybechev’s polynomial is used as basis function. Collocation and Matrix inversion method is used to derive our continuous schemes. The continuous formula is evaluated at zeros of the first Chybechev’s polynomial to give us Runge-kutta evaluation functions for the direct iteration of our solutions. Experimental examples used show that the method is A stable, highly efficient, has simple coefficients, less implementation cost when compared with similar methods in the literature.

Keywords--- Super Convergence RKM, Chybechev’s polynomial, Collocation and Matrix inversion method, Zeros of Chybechev’s polynomial, A stable

1. INTRODUCTION

Linear multi-step methods (LMM) for solving differential equations have been widely implored to solve ordinary differential equations by Lambert [1], Butcher [2] etc. These methods have often been modified to obtain better results, for examples, multi-step using Chybechev’s polynomials. (see Adeniyi et al [3], [4]), Fox [5] etc. Lie and Norsett [6] also used multi-step collocation method to develop Super convergence multi-step method for fist order ordinary differential equations (ODEs). Onumanyi et al [7] also developed new linear multi-step method with continuous coefficients for solving ODEs. This method was modified to linear multi-step implicit Runge-kutta method by Yakubu ([8],[9]), Chollom et al [10] etc. These methods mentioned above are good but they are of low order, having bigger coefficients and high computational cost when compare with our proposed method. In our paper we have addressed those problems and developed a superior Runge-kutta method with with simple coefficients, low implementation cost and more efficient than Lie et al [6].

2. METHODOLOGY

We shall find s-stage implicit Runge-kutta method of solution of first order differential equation of the form

\[ y' = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \] (2.01)

We use a polynomial of the form

\[ y(x) = \sum_{j=0}^{t-1} \alpha_j(x)y_n + h \sum_{j=1}^{m-1} \beta_j(x)f\left(\xi_j, y(\xi_j)\right) \] (2.02)

Where \( t \) denotes the number of interpolation points \( x_{n+j} \) \((j = 0, \ldots, t - 1)\), \( m \) denotes the number of collocation points, \( \xi_j \) \((j = 0, \ldots, m - 1)\) are the collocation points and \( f(x, y) \) is are smooth function.

The constants coefficients \( \alpha_j, \beta_j \) are elements of \((t + m)\times(t + m)\) square matrix. They are selected so that high accurate approximation of the solution (2.01) is obtained, \( h \) is a constant step size.

The functions \( \alpha_j(x), \beta_j(x) \) in (2.02) can be represented by polynomial of the form
\[ \alpha_j(x) = \sum_{i=0}^{t-1} \alpha_j, i + 1x^i, \quad j = (0,1, \ldots, t-1) \]

\[ h\beta_j(x) = \sum_{i=0}^{m-1} h\beta_j, i + 1x^i, \quad j = (0,1, \ldots, m-1) \]  \hfill (2.03)

The coefficients \( \alpha_j, i + 1 \) and \( \beta_j, i + 1 \) are to be determined.

Substituting (2.03) in (2.02), we have

\[ y(x) = \sum_{i=0}^{m+1-t} \left( \sum_{j=0}^{t-1} \alpha_j, i + 1y_{n+j} + h \sum_{j=0}^{m-1} h\beta_j, i + 1f_{n+j} \right) x^i = \sum_{i=0}^{m+1-t} \alpha_j, x^i \]  \hfill (2.04)

where

\[ \alpha_j = \left( \sum_{i=0}^{t-1} \alpha_j, i + 1y_{n+j} + h \sum_{j=0}^{m-1} h\beta_j, i + 1f_{n+j} \right) \]

\[ \alpha_j \in R, j \in [0,1, \ldots, t + m - 1], y \in C^m[a,b]. \]

This can be expressed in matrix form as

\[ y(x) = (y_n, \ldots, y_{n+t-1}, f_n, \ldots, f_{n+m-1}) C^T (1, x_n, \ldots, x_n^{t+m-1})^T \]

where

\[ C = \begin{bmatrix}
\alpha_{01} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\
\alpha_{02} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\
\alpha_{03} & \cdots & \alpha_{t-1,3} & h\beta_{0,3} & \cdots & h\beta_{m-1,3} \\
\alpha_{04} & \cdots & \alpha_{t-1,4} & h\beta_{0,4} & \cdots & h\beta_{m-1,4} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{0,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m}
\end{bmatrix} \]  \hfill (2.05)

\[ D = \begin{bmatrix}
1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\
0 & 1 & 2x_n & \cdots & (t + m - 1)x_n^{t+m-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 2x_n & \cdots & (t + m - 1)x_n^{t+m-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 2x_n & \cdots & (t + m - 1)x_n^{t+m-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix} \]  \hfill (2.06)

\[ q_i \text{ are the collocation points} \]

**THEOREM 1.0**

Let \( I \) denote the identity Matrix of dimension \((m + t)x (m + t)\) and Matrices \( C \) and \( D \) defined by (2.05) and (2.06) satisfies

i) \[ DC = I \]

\[ y(x) = \sum_{i=0}^{m+1-t} \left( \sum_{j=0}^{t-1} \alpha_j, i + 1y_{n+j} + \sum_{j=0}^{m-1} h\beta_j, i + 1f_{n+j} \right) x^i \] \hfill (2.07)

(see proof (Onumanyi e tal [7]))

Now we assume a power series solution of degree 5 of the form

\[ y(x) = \sum_{j=0}^{5} a_jx^j, \quad y'(x) = \sum_{j=0}^{5} ja_jx^{j-1} \]

we interpolate at \( x = x_n \), and collocate at \( x = x_{n+q_i}, i = 0,1, \ldots, 5 \)
equation (2.08) yields system of simultaneous equation of the form

\[ a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_5 x_n^5 = y_n \]
\[ a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_3 x_n^3 + 5a_5 x_n^4 = f_{n+q_1} \]
\[ a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_3 x_n^3 + 5a_5 x_n^4 = f_{n+q_2} \]
\[ \cdots \]
\[ a_1 + 2a_2 x_n + 3a_3 x_n^2 + 5a_5 x_n^4 = f_{n+q_5} \]

(2.09)

where \( a_j \) are parameters to be determined.

When (2.09) is rewritten in matrix form, we have

\[
D = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\
0 & 1 & 2x_{n+q_1} & 3x_{n+q_1}^2 & 4x_{n+q_1}^3 & 5x_{n+q_1}^4 \\
0 & 1 & 2x_{n+q_2} & 3x_{n+q_2}^2 & 4x_{n+q_2}^3 & 5x_{n+q_2}^4 \\
0 & 1 & 2x_{n+q_3} & 3x_{n+q_3}^2 & 4x_{n+q_3}^3 & 5x_{n+q_3}^4 \\
0 & 1 & 2x_{n+q_4} & 3x_{n+q_4}^2 & 4x_{n+q_4}^3 & 5x_{n+q_4}^4 \\
0 & 1 & 2x_{n+q_5} & 3x_{n+q_5}^2 & 4x_{n+q_5}^3 & 5x_{n+q_5}^4 \\
\end{bmatrix} = D \text{ matrix}
\]

\[ i.e \ DA = Y , \] where

\[ D = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\
0 & 1 & 2x_{n+q_1} & 3x_{n+q_1}^2 & 4x_{n+q_1}^3 & 5x_{n+q_1}^4 \\
0 & 1 & 2x_{n+q_2} & 3x_{n+q_2}^2 & 4x_{n+q_2}^3 & 5x_{n+q_2}^4 \\
0 & 1 & 2x_{n+q_3} & 3x_{n+q_3}^2 & 4x_{n+q_3}^3 & 5x_{n+q_3}^4 \\
0 & 1 & 2x_{n+q_4} & 3x_{n+q_4}^2 & 4x_{n+q_4}^3 & 5x_{n+q_4}^4 \\
0 & 1 & 2x_{n+q_5} & 3x_{n+q_5}^2 & 4x_{n+q_5}^3 & 5x_{n+q_5}^4 \\
\end{bmatrix}
\]

The D-matrix is non-singular and has inverse \( D^{-1} = C \)

Using Maple Mathematical software, we obtain the coefficients \( a_j \ j = (0,1,\ldots,5) \) and from (2.04) we have

\[ y(x) = a_0(x)y_n + h(\beta_{n+q_1}(x)f_{n+q_1} + \beta_{n+q_2}(x)f_{n+q_2} + \beta_{n+q_3}(x)f_{n+q_3} + \beta_{n+q_4}(x)f_{n+q_4} + \beta_{n+q_5}(x)f_{n+q_5}) \]

(2.10a)

Solving for \( a_j \) we obtain a continuous scheme for \( y(x) \).

When substituting the values of \( a_j \) and \( \beta_{n+q_i} \) \( i = 1,2,\ldots,5 \) in (2.10a) ie \( q_1 = \frac{1}{2} \), \( q_2 = \frac{1}{4} \), \( q_3 = \frac{1}{2} \), \( q_4 = \frac{3}{4} \), \( q_5 = \frac{1}{2} + \frac{\sqrt{3}}{4} \), we have the following discrete schemes.

\[ y_{n+q_1} = y_n + \left( \frac{7}{90} + \frac{3\sqrt{3}}{40} \right) h f_{n+q_1} + \left( \frac{27}{320} - \frac{3\sqrt{3}}{40} \right) h f_{n+q_2} + \left( \frac{13}{90} - \frac{\sqrt{3}}{15} \right) h f_{n+q_3} \]
\[ + \left( \frac{37}{320} - \frac{3\sqrt{3}}{40} \right) h f_{n+q_4} + \left( \frac{7}{90} - \frac{41\sqrt{3}}{960} \right) h f_{n+q_5} \]

\[ y_{n+q_2} = y_n + \left( \frac{29}{360} + \frac{3\sqrt{3}}{64} \right) h f_{n+q_1} + \left( \frac{31}{320} - \frac{1}{90} \right) h f_{n+q_2} + \frac{1}{320} h f_{n+q_4} \]

\[ y_{n+q_4} = y_n + \left( \frac{37}{320} - \frac{3\sqrt{3}}{40} \right) h f_{n+q_1} + \left( \frac{27}{320} - \frac{3\sqrt{3}}{40} \right) h f_{n+q_2} + \left( \frac{13}{90} - \frac{\sqrt{3}}{15} \right) h f_{n+q_3} + \left( \frac{7}{90} - \frac{41\sqrt{3}}{960} \right) h f_{n+q_5} \]
\begin{align*}
\frac{29}{360} + \frac{3\sqrt{3}}{64} h f_{n+q_5} + \left( \frac{7}{90} + \frac{\sqrt{3}}{24} \right) h f_{n+q_1} + \frac{9}{40} h f_{n+q_2} + \frac{13}{90} h f_{n+q_3} - \frac{1}{40} h f_{n+q_4} + \left( \frac{7}{90} - \frac{\sqrt{3}}{24} \right) h f_{n+q_5} \\
y_{n+q_3} = y_n + \left[ \frac{3}{40} + \frac{3\sqrt{3}}{64} \right] h f_{n+q_1} + \frac{63}{320} h f_{n+q_2} + \frac{3}{10} h f_{n+q_3} + \frac{33}{320} h f_{n+q_4} + \left( \frac{3}{40} - \frac{3\sqrt{3}}{64} \right) h f_{n+q_5} \\
y_{n+q_4} = y_n + \left[ \frac{7}{90} + \frac{41\sqrt{3}}{960} \right] h f_{n+q_1} + \left[ \frac{27}{320} + \frac{3\sqrt{3}}{40} \right] h f_{n+q_2} + \left( \frac{13}{90} + \frac{\sqrt{3}}{15} \right) h f_{n+q_3} + \frac{7}{90} - \frac{3\sqrt{3}}{320} h f_{n+q_5} \\
y_{n+q_5} = y_n + \frac{7}{90} + \frac{\sqrt{3}}{24} h f_{n+q_1} + \frac{27}{320} + \frac{3\sqrt{3}}{40} h f_{n+q_2} + \left( \frac{13}{90} + \frac{\sqrt{3}}{15} \right) h f_{n+q_3} + \frac{7}{90} - \frac{3\sqrt{3}}{320} h f_{n+q_5} \\
\text{To convert to Runge-Kutta, the discrete schemes must satisfy the differential equation (2.01), that is}
\end{align*}

\begin{align*}
y'_{n+q_1} &= f(x_{n+q_1}, y_{n+q_1}) = f(x_n, y_n) + \left( \frac{7}{90} + \frac{3\sqrt{3}}{320} \right) h f_{n+q_1} + \left( \frac{27}{320} - \frac{3\sqrt{3}}{40} \right) h f_{n+q_2} \\
&+ \left( \frac{13}{90} - \frac{\sqrt{3}}{15} \right) h f_{n+q_3} + \left[ \frac{37}{320} + \frac{3\sqrt{3}}{40} \right] h f_{n+q_4} + \left( \frac{7}{90} - \frac{\sqrt{3}}{24} \right) h f_{n+q_5} \\
y'_{n+q_2} &= f(x_{n+q_2}, y_{n+q_2}) = f(x_n, y_n) + \left( \frac{29}{360} + \frac{3\sqrt{3}}{64} \right) h f_{n+q_1} + \frac{63}{320} h f_{n+q_2} + \frac{3}{10} h f_{n+q_3} + \frac{33}{320} h f_{n+q_4} + \left( \frac{3}{40} - \frac{3\sqrt{3}}{64} \right) h f_{n+q_5} \\
y'_{n+q_3} &= f(x_{n+q_3}, y_{n+q_3}) = f(x_n, y_n) + \left( \frac{7}{90} + \frac{\sqrt{3}}{24} \right) h f_{n+q_1} + \frac{9}{40} h f_{n+q_2} + \frac{13}{90} h f_{n+q_3} - \frac{1}{40} h f_{n+q_4} + \left( \frac{7}{90} - \frac{\sqrt{3}}{24} \right) h f_{n+q_5} \\
y'_{n+q_4} &= f(x_{n+q_4}, y_{n+q_4}) = f(x_n, y_n) + \left( \frac{3}{40} + \frac{3\sqrt{3}}{64} \right) h f_{n+q_1} + \frac{63}{320} h f_{n+q_2} + \frac{3}{10} h f_{n+q_3} + \frac{33}{320} h f_{n+q_4} + \left( \frac{3}{40} - \frac{3\sqrt{3}}{64} \right) h f_{n+q_5} \\
y'_{n+q_5} &= f(x_{n+q_5}, y_{n+q_5}) = f(x_n, y_n) + \frac{7}{90} + \frac{\sqrt{3}}{24} h f_{n+q_1} + \frac{27}{320} + \frac{3\sqrt{3}}{40} h f_{n+q_2} + \frac{13}{90} + \frac{\sqrt{3}}{15} h f_{n+q_3} + \frac{7}{90} - \frac{3\sqrt{3}}{320} h f_{n+q_5} \\
\text{Substituting,} \quad k_1 = f(x_n, y_{n+q_1}) = f_{n+q_1}, \quad k_2 = f(x_n, y_{n+q_2}) = f_{n+q_2} \quad \text{etc.}
\end{align*}

we obtain the following function evaluations:

\begin{align*}
k_1 &= f(x_n, y_n) + \left( \frac{7}{90} + \frac{3\sqrt{3}}{320} \right) h k_1 + \left( \frac{27}{320} - \frac{3\sqrt{3}}{40} \right) h k_2 + \left( \frac{13}{90} - \frac{\sqrt{3}}{15} \right) h k_3 + \left( \frac{37}{320} - \frac{3\sqrt{3}}{40} \right) h k_4 \\
&+ \left( \frac{7}{90} - \frac{\sqrt{3}}{24} \right) h k_5 \\
k_2 &= f(x_n, y_n) + \left( \frac{29}{360} + \frac{3\sqrt{3}}{64} \right) h k_1 + \frac{63}{320} h k_2 + \frac{3}{10} h k_3 + \frac{33}{320} h k_4 + \left( \frac{3}{40} - \frac{3\sqrt{3}}{64} \right) h k_5 \\
k_3 &= f(x_n, y_n) + \left( \frac{7}{90} + \frac{\sqrt{3}}{24} \right) h k_1 + \frac{9}{40} h k_2 + \frac{13}{90} h k_3 - \frac{1}{40} h k_4 + \left( \frac{7}{90} - \frac{\sqrt{3}}{24} \right) h k_5 \\
k_4 &= f(x_n, y_n) + \frac{3}{40} h k_1 + \frac{63}{320} h k_2 + \frac{3}{10} h k_3 + \frac{33}{320} h k_4 + \left( \frac{3}{40} - \frac{3\sqrt{3}}{64} \right) h k_5 \\
\end{align*}
\[ k_5 = f[x_n, y_n] + \left( \frac{7}{90} + \frac{41\sqrt{3}}{960} \right) hk_1 + \left( \frac{27}{320} + \frac{3\sqrt{3}}{40} \right) hk_2 + \left( \frac{13}{90} + \frac{\sqrt{3}}{15} \right) hk_3 + \left( \frac{37}{320} + \frac{3\sqrt{3}}{40} \right) hk_4 + \left( \frac{7}{90} - \frac{3\sqrt{3}}{320} \right) hk_5 \]  

(2.13)

The Butcher coefficient tableau for the above function evaluations can be written in the form below.

<table>
<thead>
<tr>
<th>C</th>
<th>(A)</th>
<th>(U)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} ) ( \frac{\sqrt{3}}{4} )</td>
<td>( \frac{7}{90} + \frac{3\sqrt{3}}{320} )</td>
<td>( \frac{13}{90} - \frac{\sqrt{3}}{40} )</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>( \frac{29}{360} + \frac{3\sqrt{3}}{64} )</td>
<td>( \frac{1}{320} )</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( \frac{7}{90} + \frac{\sqrt{3}}{64} )</td>
<td>( \frac{9}{40} )</td>
</tr>
<tr>
<td>( \frac{3}{4} )</td>
<td>( \frac{3}{40} + \frac{3\sqrt{3}}{64} )</td>
<td>( \frac{63}{320} )</td>
</tr>
<tr>
<td>( \frac{1 + \sqrt{3}}{2} )</td>
<td>( \frac{7}{90} + \frac{41\sqrt{3}}{960} )</td>
<td>( \frac{27}{320} + \frac{3\sqrt{3}}{40} )</td>
</tr>
</tbody>
</table>

| b_i    | \( \frac{7}{45} \) | \( \frac{1}{5} \) | \( \frac{13}{45} \) | \( \frac{1}{5} \) | \( \frac{7}{45} \) | 1 |

(B)

The general R-K for this method is defined as

\[ y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 \]

where

\[ b_1 = \frac{7}{45}, \quad b_2 = \frac{1}{5}, \quad b_3 = \frac{13}{45}, \quad b_4 = \frac{1}{5}, \quad b_5 = \frac{7}{45} \]

and \( k_i \) defined as (2.13).

Note that the values of \( b_i, i = 1, 2, \ldots, 5 \) are obtained by evaluating the continuous scheme (2.10a) at \( x = x_{n+1} \).

The above table can be summarized as

\[
\begin{array}{ccc}
A & U & V \\
B & & \\
\end{array}
\]

where

\[ U = (1,1,1,1,1)^T, \quad V = (1), \quad A = (a_{ij}), \quad a_{ij} \text{ are the coefficient of the Butcher table and } B = b_i, i = 1, \ldots, 5. \]

3. ANALYSIS OF THE SCHEME

(i) The exact solution is \( y(x_{n+1}) \) and Runge-kutta solution is \( (y_{n+1}) \) and the error of the method is \( E_r = y(x_{n+1}) - y_{n+1} \) and absolute error is \( AE_r = |y(x_{n+1}) - y_{n+1}| \).

The Order of the method is \( p \), where

\[ y(x_{n+1}) - y_{n+1} = C_{p+1} y^{(n+1)}(x) + C_{p+2} y^{(n+2)}(x) + \cdots + 0 h^{(p+1)}, \]

\[ c_1 = c_1 = c_1 = c_1 = \ldots = c_p = 0 \text{ and } c_{p+1} \neq 0, \quad c_{p+1} \text{ is the error constant see Butcher [2].} \]

(ii) Consistency

The Runge-kutta method is consistent because

\[ \sum_{j=0}^{5} a_{ij} = c_i, \quad \sum_{i=0}^{5} b_i = 1 \]

see Butcher Table of (214).

(iii) Stability

The stability function \( \Re(z) \) is defined as
\[ R(z) = I + ZB (I - ZA)^{-1} = e \]

\[ e = (1,1,1,1,1), \quad Z = \lambda h, \quad A \text{ is the coefficient matrix, I identity} \]

\[ B = (b_1, b_2, b_3, b_4, b_5), \text{ weights.} \]

The A-stability Region is the set of points satisfying

\[ (R(z)) = \{z: |R_e(z)| < 1\} \text{ and the A-stable is the region} \]

\[ (R(z)) = \{z: R(z) \leq 0 \text{ and } |R(z)| \leq 1\} \]

The characteristic polynomial is

\[ P(z) = \det(R(z) - \lambda I). \]

The region of convergence can be plotted using mat lap etc

(iv) The exact solution at \( x = x_i \) is defined as \( y(x) \) and the approximate solution is \( x = x_i \) is \( y_i \).

Using Taylor’s series expansion

\[ y(x_{n+1}) = y_0 + \frac{n}{1!} h y'(x_i) + \frac{n(n-1)h^2}{2!} y''(x_i) + \cdots + \frac{n(n-1)(n-2)h^{n-1}}{n!} y^{(n-1)}(x_i) \]

and

\[ y_{n+1} = y_n + \frac{7}{45} h k_1 + \frac{1}{5} h k_2 + \frac{13}{45} h k_3 + \frac{1}{5} h k_4 + \frac{7}{45} h k_5 \]

The order of our scheme is \( p = 6 \), since \( c_0 = c_1 = \cdots = c_6 = 0 \) and \( c_{p+1} = c_7 = \frac{1}{8601600} \) error constant.

4. NUMERICAL EXPERIMENTS

We use similar methods developed by lie S Norsett[11] and Adegboye and Yahaya [12] to observe the level of performances and efficiency of our new method in the following problems:

**Problem 1**

\[ y' = 8(x - y) + 1, \quad y(0) = 2, \quad h = 0.02 \]

Exact solution: \( y(x) = x + 2e^{-8x} \)

**Problem 2**

\[ y' = \sin x - 2y, \quad y\left(\frac{\pi}{2}\right) = 3, \quad h = \frac{\pi}{50} \]

Exact solution: \( y(x) = \frac{1}{10} \cos x + \frac{3}{10} \sin x + \frac{27}{10} e^{-\frac{3x}{2}} \)

**Problem 3**

\[ y' + 20y = 20x^2 + 2x, \quad y(0) = \frac{1}{3}, \quad h = 0.025 \]

Exact solution: \( y(x) = x^2 + \frac{1}{3} e^{-20x} \)

The following are solution tables for the given problems above.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>Present method of (2.14)</th>
<th>Error of method [10]</th>
<th>Error of Present method (2.14)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>1.724287577932420</td>
<td>1.724287577929230</td>
<td>2.56 E-11</td>
<td>3.19 E-12</td>
</tr>
<tr>
<td>0.04</td>
<td>1.492298074147380</td>
<td>1.492298074141940</td>
<td>4.35 E-11</td>
<td>5.44 E-12</td>
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<tr>
<td>0.06</td>
<td>1.297566783612280</td>
<td>1.297566783556630</td>
<td>5.57 E-11</td>
<td>6.96 E-12</td>
</tr>
<tr>
<td>0.08</td>
<td>1.134584848086100</td>
<td>1.134584848078190</td>
<td>6.32 E-11</td>
<td>7.91 E-12</td>
</tr>
<tr>
<td>0.10</td>
<td>0.998657928234444</td>
<td>0.998657928226020</td>
<td>6.74 E-11</td>
<td>8.42 E-12</td>
</tr>
</tbody>
</table>
Table 2: Comparison of Numerical solution of Problem 2

<table>
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</thead>
<tbody>
<tr>
<td>$\frac{13\pi}{25}$</td>
<td>2.5418383600099</td>
<td>2.541838359996723</td>
<td>1.05 E -10</td>
<td>1.32 E -11</td>
</tr>
<tr>
<td>$\frac{27\pi}{50}$</td>
<td>2.162157581773023</td>
<td>2.162157581751139</td>
<td>1.75 E -10</td>
<td>2.19 E -11</td>
</tr>
<tr>
<td>$\frac{14\pi}{25}$</td>
<td>1.847250042547062</td>
<td>1.847250042519876</td>
<td>2.17 E -10</td>
<td>2.72 E -11</td>
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<tr>
<td>$\frac{29\pi}{50}$</td>
<td>1.585764824898583</td>
<td>1.585764824898583</td>
<td>2.40 E -10</td>
<td>3.00 E -11</td>
</tr>
<tr>
<td>$\frac{3\pi}{5}$</td>
<td>1.36830725239477</td>
<td>1.36830725208400</td>
<td>2.48 E -10</td>
<td>3.10 E -11</td>
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</tbody>
</table>

Table 3: Comparison of Numerical solution of Problem 3

<table>
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<tr>
<td>0.5</td>
<td>0.125126480390481</td>
<td>0.12512647903</td>
<td>1.97 E -09</td>
<td>1.36 E -09</td>
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<tr>
<td>0.1</td>
<td>0.055111761078871</td>
<td>0.0551117600813</td>
<td>1.30 E -08</td>
<td>9.98 E -10</td>
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<tr>
<td>0.15</td>
<td>0.039095689455955</td>
<td>0.0390956889055</td>
<td>6.61 E -09</td>
<td>5.50 E -10</td>
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<tr>
<td>0.2</td>
<td>0.046105212962911</td>
<td>0.0461052126929</td>
<td>3.93 E -09</td>
<td>2.70 E -10</td>
</tr>
</tbody>
</table>

Note: The Method [11] has 5 iterations before obtaining the next solution while present method [2.14] has only 2 iterations.

5. CONCLUSION

We developed a highly super convergent Runge-kutta method for solution of first order order ODEs. The method is more superior than Lie and Norsett methods (see numerical Examples). It has lesser implementation cost.(see example3). For similar method mentioned in this research work you need to iterate five times before getting to the next solution (see Yahaya and Adegoke [11]). In our new method only two iterations is needed before getting the next solution and the results obtained are more efficient and stable than methods ([10], [11]). Finally we have successfully developed a new super convergent implicit RK method which converges perfectly more than the existing methods.

6. REFERENCES

2. Butcher JC (1996) general linear methods, Computational Mathematics and Application, volume (31)4,5 pages 105-112
5. Fox L and Parker IB (1968): “Chebyshev polynomial in Numerical Analysis” Addison-Wesley Philippines


