The Extended Riesz Theorem and its Results

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ABSTRACT— The main purpose of this paper is to extended the Riesz theorem in fuzzy anti $n$-normed linear spaces as a generalization of linear $n$-normed space. Also we study some properties of fuzzy anti $n$-normed linear spaces.

Keywords— Riesz theorem, Fuzzy $n$-compact sets, Fuzzy anti $n$-norms, $\alpha$ $n$-norms.

1. INTRODUCTION

A satisfactory theory of 2 norms of a linear space has been introduced and developed by Gahler to $n$-norm on a linear space [6]. In following H. Gunawan and M. Mashadi [7], S. S. Kim and Y. J. Cho [11], R. Malceski [17] and A. Misiak [18] developed the theory of $n$-normed space [18]. The more details about the theory of fuzzy normed linear space can be found in [1, 2, 5, 21]. The concept of fuzzy sets was introduced by L. A. Zadeh in 1965 [26] and thereafter several authors applied it in different branches of pure and applied Mathematics. The concept of fuzzy norms was introduced by A. K. Katsaras in 1984 [9]. In 1992, C. Felbin introduced the concept of Fuzzy normed linear space[5]. The notion of Fuzzy 2 normed linear spaces introduced by A.R. Meenakshi and R. Gokilavani in 2001. B. Sundander Reddy introduced the idea of Fuzzy anti 2-normed linear spaces [25]. AL. Narayanan and S. Vijayabalaji introduced the definition of fuzzy anti $n$-norm on a linear space and Also, Vijayabalaji [19] and Thillaigovindan introduced study of the complete fuzzy $n$-normed linear spaces [27]. I. H. Jebril and S. K. Samanta gave the definition of a Fuzzy anti normed linear space in 2011 [16]. F. Riesz obtained the Riesz theorem in a normed space[22]. Park and Chu have extended the Riesz theorem in a normed space to $n$-normed linear space [20].

Following Kavikumar, Yang Bae Jun and Azme Khamis [10], in this paper extend the Riesz theorem in $n$-normed linear spaces to fuzzy Anti $n$-normed linear spaces. Also, we establish some basic results.

2. PRIMILINARIES

The main purpose of this article is the extension of Riesz theorem to fuzzy anti $n$-normed linear spaces. In the first part, we try to establish some basic theorems and by aimes of this result, we do our main goal.

Definition 2.1 [8] If $W$ is a linear subspace of a finite-dimentional vector space $V$, then the codimension of $W$ in $V$ is the difference between the dimensions,

$$\text{codim}(W) = \dim(V) - \dim(W)$$

Definition 2.2 [10] Let $n \in \mathbb{N}$ and $X$ be a real linear space of dimension $d \geq n$. (Here we allow $d$ to be infinite). A real valued function $\| \cdot, \ldots, \cdot \|$ on $X \times \cdots \times X$ ($n$ times $= X^n$) satisfying four properties:

(N1) $\| x_1, \ldots, x_n \| = 0$ iff $x_1, \ldots, x_n$ are linearly dependent,

(N2) $\| x_1, \ldots, x_n \|$ is invariant under any permutation of $x_1, \ldots, x_n$,

(N3) $\| x_1, \ldots, cx_n \| = |c| \| x_1, \ldots, x_n \|$, for any real $c$,

(N4) $\| x_1, \ldots, x_{n-1}, y + z \| \leq \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \|$, is called a $n$-normed on $X$ and the pair $(X, \| \cdot, \ldots, \cdot \|)$ is called a $n$-normed linear space.
**Definition 2.3** [10] A sequence \( \{x_n\} \) in a linear n-normed space \((X, \alpha, \ldots, \alpha)\) is said to be n-convergent to \( x \in X \) and denote by \( x_n \rightarrow x \) as \( k \rightarrow \infty \) if
\[
\lim_{k \to \infty} \|x_1, \ldots, x_{n-k}, x_n - x\| = 0.
\]

**Definition 2.4** [15] A subset of a linear n-normed space \((X, \alpha, \ldots, \alpha)\) is called a n-compact proper subset if for every sequence \( \{x_n\} \) in \( Y \), there exists a subsequence of \( \{x_n\} \) which converges to an element \( x \in X \).

From this viewpoint, Park and Chu [20] obtained the following theorem in n-normed spaces:

**Theorem 2.1** [10] Let \( Y \) and \( Z \) be two subspaces of a linear n-normed space \( X \), and \( Y \) be a n-compact proper subset of \( Z \) with codimension greater than \( n - 1 \). For each \( \theta \in (0, 1) \), there exists an element \( (z_1, \ldots, z_n) \in Z_n \) such that
\[
\|z_1, \ldots, z_n\| = 1, \quad \|z_1 - y, \ldots, z_n - y\| \geq \theta,
\]
for all \( y \in Y \).

**Definition 2.5** [3] A binary operation \( \odot : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous t-conorm if \( \odot \) satisfies the following conditions:

(i) \( \odot \) is commutative and associative,
(ii) \( \odot \) is continuous,
(iii) \( \alpha \odot \alpha = \alpha \), \( \forall \alpha \in [0, 1] \),
(iv) \( \alpha \odot \beta \leq \alpha \odot \odot \) whenever \( \alpha \leq \beta \), \( \forall \alpha, \beta, \gamma \in [0, 1] \).

A few examples of continuous t-conorm are \( \odot = a + b - ab, \odot = \max\{a, b\} \) and \( \odot = \min\{a, b\} \).

**Remark 2.1** [1] For any \( a, b \in (0, 1) \) with \( a > b \) there exists \( c \in (0, 1) \) such that \( a > c \odot \).

**Definition 2.6** [27] Let \( X \) be a linear space over a real field \( F \). A fuzzy subset \( N \) of \( X^* \times [0, \infty) \) is called a fuzzy anti-n-norm on \( X \) if and only if:

(\( FAN1 \)) for all \( t \in \mathbb{R} \) with \( t \leq 0 \), \( N(x_1, \ldots, x_n, t) = 1 \).

(\( FAN2 \)) for all \( t \in \mathbb{R} \) with \( t > 0 \), \( N(x_1, \ldots, x_n, t) = 1 \), \( x_1, \ldots, x_n \) are linearly dependent,

(\( FAN3 \)) \( N(x_1, \ldots, x_n, t) \) is invariant under any permutation of \( x_1, \ldots, x_n \),

(\( FAN4 \)) \( N(x_1, \ldots, x_n, t) = N(x_1, \ldots, x_n, t|c) \) if \( c \neq 0, c \in F \),

(\( FAN5 \)) \( N(x_1, \ldots, x_n + x_{n+1}, s+t) \leq N(x_1, \ldots, x_n, s) \odot N(x_1, \ldots, x_n, t) \) for all \( s, t \in \mathbb{R} \),

(\( FAN6 \)) \( N(x_1, \ldots, x_n, t) \) is a continuous and non-increasing function of \( t \) such that
\[
\lim_{t \to \infty} N(x_1, \ldots, x_n, t) = 0.
\]

Then \((X, N)\) is called a fuzzy anti-n-normed linear space.

**Definition 2.7** [27] A sequence \( \{x_n\} \) in a fuzzy anti-n-normed space \((X, N)\) is said to converge to \( x \) if for given \( r > 0, t > 0 \) and \( 0 < r < 1 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, \ldots, x_{n-k}, x_n - x, t) < r \), for all \( n \geq n_0 \).

**Example 2.1** [27] Let \((X, \|\cdot, \ldots, \cdot\|)\) be a n-normed linear space. Define,
\[
N(x_1, \ldots, x_n, t) = \begin{cases} 1 - \frac{t}{\|x_1, \ldots, x_n\|} & t > 0, \forall x \in X, \\ 1 & t \leq 0, \forall x \in X. \end{cases}
\]

Then \((X, N)\) is a fuzzy anti-n-normed linear space.

**Theorem 2.2** [27] Let \((X, N)\) be a fuzzy anti-n-normed space. Assume that condition that

(\( FAN7 \)) \( N(x_1, \ldots, x_n, t) > 0, \forall t > 0 \), implies \( x_1, \ldots, x_n \) are linearly dependent. Define \( \|x_1, \ldots, x_n\| = \sup\{t : N(x_1, \ldots, x_n, t) \leq 1 - \alpha\}, \alpha \in (0, 1) \). Then \( \{\|\cdot, \ldots, \cdot\| ; \alpha \in (0, 1)\} \) is a descending family of \( n \)-norms on \( X \). These \( n \)-norms are called \( \alpha \)-\( n \)-norms on \( X \) corresponding to the fuzzy anti-\( n \)-norm on \( X \).

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Definition 2.8 [2] The fuzzy normed space \((X, N)\) is said to be a fuzzy anti \(n\)-normed Banach space whenever \(X\) is complete with respect to the fuzzy metric induced by fuzzy anti \(n\)-norm.

3. FUZZY RIESZ THEOREM

Riesz [22] obtained the following theorem in a normed space.

Theorem 3.1 [22] Let \(Y\) and \(Z\) be subspaces of a normed space \(X\), and \(Y\) a closed proper subset of \(Z\). For each \(\theta \in (0, 1)\), there exists an element \(z \in Z\) such that

\[
\|z\| = 1, \quad \|z - y\| \geq \theta,
\]

for all \(y \in Y\).

Now we try to extend Riesz theorem to fuzzy anti \(n\)-normed linear spaces. Also, we prove some corollaries of this theorem at the end of this section.

Definition 3.1 A subset \(Y\) of a fuzzy anti \(n\)-normed linear space \((X, N)\) is called a fuzzy \(n\)-compact subset if for every sequence \(\{y_n\}\) in \(Y\), there exists a subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) which converges to an element \(y \in Y\). In other words, given \(t > 0\) and \(0 < r < 1\), there exists an integer \(n_0 \in \mathbb{N}\) such that

\[
N(y_{n_k}, y - y, t/k) < r,
\]

for all \(n, k \geq n_0\) and \(n_j \geq n_0\).

Lemma 3.1 Let \((X, N)\) be a fuzzy anti \(n\)-normed linear space. Assume that \(x_i \in X\) for each \(i \in \{1, \ldots, n\}\) and \(c \in F\). Then

\[
N(x_1, x_2 + cx_1 + x_n, t) = N(x_1, x_2 + cx_1 + x_n, t)
\]

Proof. 

\[
N(x_1, x_2 + cx_1 + x_n, t) = N(x_1, x_2 + cx_1 + x_n, t/2 + t/2) \\
\leq \max\{N(x_1, x_2 + cx_1 + x_n, t/2), N(x_1, x_2 + cx_1 + x_n, t/2)\} \\
= \max\{N(x_1, x_2 + cx_1 + x_n, t/2), N(x_1, x_2 + cx_1 + x_n, t/2)\} \\
= \max\{N(x_1, x_2 + cx_1 + x_n, t/2), N(x_1, x_2 + cx_1 + x_n, t/2)\} \\
\geq N(x_1, x_2 + cx_1 + x_n, t).
\]

\[
\square
\]

Theorem 3.2 Let \((X, N)\) be a fuzzy anti \(n\)-normed linear space. If the

\[
sup\{t > 0 : N(x_1, x_2, x_n - y, t)\} = 0,
\]

for \((x_1, \ldots, x_n) \in X^n\) and \(Y\) is a fuzzy \(n\)-compact subset of \(X\), then there exists an element \(y_0 \in Y\) such that

\[
\{t > 0 : N(x_1, x_2, x_n - y_0, t)\} = 0,
\]

Proof. Let \(t > 0\) and \(\varepsilon \in (0, 1)\). Choose \(r \in (0, 1)\) such that \(r^2r < \varepsilon\) (remark 2.1). Since \(Y\) is a fuzzy \(n\)-compact subset of \(X\), there exists an integer \(n_0 \in \mathbb{N}\) such that

\[
N(x_1, x_2, x_n - y_0, ct) < r,
\]

for all \(n, k \geq n_0\) and a constant \(c\). Since \(\{y_k\}\) is a sequence in a fuzzy \(n\)-compact subset \(Y\) of \(X\). Without loss of generality assume that \(\{y_k\}\) converges to \(y_0 \in Y\), as \(k \to \infty\). Then for given, \(0 < \lambda < 1\), there exists an integer \(n_1 \in \mathbb{N}\) such that

\[
N(y_k - y_0, \omega_2, \ldots, \omega_n, t) < \lambda,
\]

for all \(\omega_j \in X (j = 1, \ldots, n)\) and \(n_0 > n_1\). For every \(r \in (0, 1)\), there exists \(\lambda \in (0, 1)\) such that (remark 2.1)
by lemma 3.1, if \( n_0 > n_1 \), then we have

\[
N(x_1 - y_0, x_2 - y_0, \ldots, x_n - y_0, t) \leq N(y_k - y_0, x_2 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_0, \ldots, x_n - y_0, (k-1)t/k)
\]

\[
\leq N(y_k - y_0, x_2 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3 - y_k, x_4 - y_0, \ldots, x_n - y_0, (k-2)t/k)
\]

\[
\leq N(y_k - y_0, x_2 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3, x_4 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, y_k - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3 - y_k, x_4 - y_0, \ldots, x_n - y_0, (k-3)t/k)
\]

\[
\leq N(y_k - y_0, x_2 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, y_k - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3 - y_k, y_k - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta \cdots
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3 - y_k, x_4 - y_0, y_k - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3 - y_k, x_4 - y_k, \ldots, x_{n-1} - y_k, x_n - y_0, (k-(n-1)t/k)
\]

Therefore

\[
N(x_1 - y_0, x_2 - y_0, \ldots, x_n - y_0, t) \leq N(y_k - y_0, x_2 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, y_k - y_0, x_3 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, y_k - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta \cdots
\]

\[
\Delta N(x_1 - y_k, x_2 - y_k, x_3 - y_k, x_4 - y_k, \ldots, x_{n-1} - y_k, y_k - y_0, (k-n)t/k)
\]

\[
= N(y_k - y_0, x_2 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_k, x_2 - y_0, x_3 - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta N(x_1 - y_0, x_2 - y_0, y_k - y_0, \ldots, x_n - y_0, t/k)
\]

\[
\Delta \cdots
\]

\[
\Delta N(x_1 - y_0, x_2 - y_0, x_3 - y_0, \ldots, x_{n-1} - y_0, y_k - y_0, t/k)
\]

\[
\Delta N(x_1 - y_0, x_2 - y_0, x_3 - y_0, \ldots, x_{n-1} - y_0, x_n - y_0, ct)
\]

\[
\overset{n}{\lambda^0 \lambda^0 \cdots \lambda^0 r} < r,
\]
< r \r < \varepsilon.

Since \varepsilon is arbitrary,

\[ \sup\{ t > 0 : N(x_i - y_0, x_2 - y_0, \ldots, x_n - y_0, t) \} = 0 \]

Now, we represent Riesz Theorem for fuzzy anti n-normed linear spaces.

**Theorem 3.3 Riesz Theorem.** Let \((X, N)\) be a fuzzy anti n-normed linear space satisfying condition \((FAN7)\) and \([\| \cdot, \cdot \| : \alpha \in (0,1)]\) be a descending family of \(\alpha\)-n-norms corresponding to \((X, N)\). Let \(Y\) and \(Z\) be subspaces of \(X\) and \(Y\) be a fuzzy n-compact proper subset of \(Z\) with \(\dim Z \geq n\). For each \(k \in (0,1)\), there exists an element \((z_1, \ldots, z_n) \in Z_n\) such that

\[ \| z_1, \ldots, z_n \|_\alpha = 0, \quad N(z_1 - y, \ldots, z_n - y, t) \geq \alpha, \]

for all \(y \in Y\).

**Proof.** Let \(\alpha \in (0,1)\). \((v_1, \ldots, v_n) \in Z - Y\) with \(v_1, \ldots, v_n\) are linearly independent. Let

\[ \sup_{y \in Y} \| v_1 - y, \ldots, v_n - y \|_\alpha = k. \]

We follow the proof in two cases:

**Case (i):** Assume that \(k = 0\). By theorem 3.2, there is an element \(y_0 \in Y\) such that \(N(v_1 - y_0, \ldots, v_n - y_0) = 0\).

If \(y_0 = 0\), then \(v_1, \ldots, v_n\) are linearly independent, which is a contradiction.

If \(y_0 \neq 0\), then \(v_1, \ldots, v_n\) are linearly independent.

**Case (ii):** Let \(k > 0\), where

\[ k = \| v_1 - y, \ldots, v_n - y \|_\alpha = \sup\{ s : N(v_1 - y, \ldots, v_n - y, s) \leq \alpha \}. \]

Since \(N(v_1 - y, \ldots, v_n - y, s)\) is continuous (definition 2.6), now we have (by theorem 4.4, in [19]),

\[ N(v_1 - y, \ldots, v_n - y, s) \leq 1 - \alpha, \]

So for each \(k_i \in (0,1)\), there exists an element \(y_0 \in Y\) such that

\[ k \geq \| v_1 - y_0, \ldots, v_n - y_0 \|_\alpha \geq \frac{k}{k_i}. \]

For each \(j = 1, \ldots, n\), let

\[ z_j = \frac{v_j - y_0}{\| v_1 - y_0, \ldots, v_n - y_0 \|_\alpha}. \]

Then it is obvious that \( \| z_1, \ldots, z_n \|_\alpha = 0 \).

Now,

\[
\| z_1 - y_0, \ldots, z_n - y_0 \|_\alpha = \left\| \frac{v_1 - y_0}{\| v_1 - y_0, \ldots, v_n - y_0 \|_\alpha} - y, \ldots, \frac{v_n - y_0}{\| v_1 - y_0, \ldots, v_n - y_0 \|_\alpha} - y \right\|_\alpha \\
= \left\| \frac{1}{\| v_1 - y_0, \ldots, v_n - y_0 \|_\alpha} \| v_1 - (y_0 + y) \| v_1 - y_0, \ldots, v_n - y_0 \|_\alpha \right\|_\alpha \\
= \| v_n - (y_0 + y) \| v_1 - y_0, \ldots, v_n - y_0 \|_\alpha \]
By (FAN7), there exists $\alpha \in (0,1)$ such that

$$\sup\{k > 0: N(z_1 - y, \ldots, z_n - y, k) \leq 1 - \alpha\} \leq k_1.$$ 

Then there exists $\alpha_0 \in (0,1)$ such that

$$N(z_1 - y, \ldots, z_n - y, k_1) > \alpha_0 \geq 1 - \alpha,$$

for all $y \in Y$.

**Corollary 3.1** Given a strictly nested sequence of closed subspaces

$$\{0\} \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq N_4 \subseteq \ldots$$

of a fuzzy Banach space $X$, one can find a sequence of vectors $x_1, \ldots, x_n \in N_n$ with $\|x_1, \ldots, x_n\|_0 = 0$ and $N(x_i - N_n, \ldots, x_n - N_n) \geq \frac{1}{2}$. Similarly, for a sequence of closed subspaces nested in the opposite direction $\{0\} \subseteq R_1 \subseteq R_2 \subseteq R_3 \subseteq R_4 \subseteq \ldots$, there are unit vectors $x_n \in R_n$ with $N(x_1 - R_n, \ldots, x_n - R_n) > \frac{1}{2}$.

**Proof.** Pick any $x_1$ of norm $N_1$. Let $F_1$ be the linear span of $x_1$. Then $F_1$ is finite dimensional and, hence, closed. By Riesz’s Lemma, there is an $x_2$ of norm $N_2$ such that $N(x_1 - N_1, x_2 - N_1) \geq \frac{1}{2}$. Let $F_2$ be the linear span of $x_1$ and $x_2$. Then $F_2$ is finite dimensional and, hence, closed. By Riesz’s Lemma, there is an $x_3$ of norm $N_3$ such that $N(x_1 - N_1, x_2 - N_2) \geq \frac{1}{2}$. Continue ... □

The same corollary has been achieved in linear normed space as follows:

**Corollary 3.2** Given a strictly nested sequence of closed subspaces

$$\{0\} \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq N_4 \subseteq \ldots$$

of a Banach space $X$, one can find a sequence of vectors $x_n \in N_n$ with $\|x_n\| = 1$ and $\text{dist}(x_n, N_{n+1}) \geq \frac{1}{2}$. Similarly, for a sequence of closed subspaces nested in the opposite direction, $\{0\} \subseteq R_1 \subseteq R_2 \subseteq R_3 \subseteq \ldots$, there are unit vectors $x_n \in R_n$ with $\text{dist}(x_n, R_{n+1}) \geq \frac{1}{2}$.

4. REFERENCES


