Mechanical Model and Design Guideline of Wind Powered Car and Ship Using Rotary Wings Advancing at Super Wind velocity in Tailwind

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ABSTRACT— ***DDWFTTW (Dead-DownWind Faster Than The Wind) refers to a wind powered car using rotary wings that runs faster than the wind in tailwind. It was found experimentally by Jack Goodman and was experimentally demonstrated in a nearly perfect form by Rick Cavallaro. A full scale model called Blackbird with weight 200kg and propeller diameter 5m has run at the 2.8 times faster speed than tailwind. On the other hand, the theoretical verification was given by Mark Drela. The absorbed energy through rotation of wheel or water turbine is transferred to air propeller, and the thrust is generated by the air propeller. If the running resistance of the car or ship is sufficiently smaller than the thrust, the car or the ship can advance faster than the tailwind speed. The present paper discusses further using the momentum theory of rotary turbine and propeller and made the subject clearer. Especially, it clarified the problem and direction when applied to a wave powered ship. A reasonable application would bring us a big energy saving effect.***

Keywords— Wind powered car or ship, Tailwind, Faster than wind, DDWFTTW

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1. INTRODUCTION

DDWFTTW (Dead-DownWind Faster Than The Wind) refers to a wind powered car using rotary wings that runs faster than the wind in tailwind [1]. It was found experimentally by Jack Goodman in 2003 [2] and invoked a big dispute. Many people call it a quack or nonsense [3,4]. However, Rick Cavallaro demonstrated experimentally in a nearly perfect form [5]. On July 2010, a test was conducted at El Mirage Dry LakeBed in south California. A full scale model called Blackbird with weight 200kg and propeller diameter 5m has run at the 2.8 times faster speed than tailwind [3].

Figure 1: Goodman's model [2]

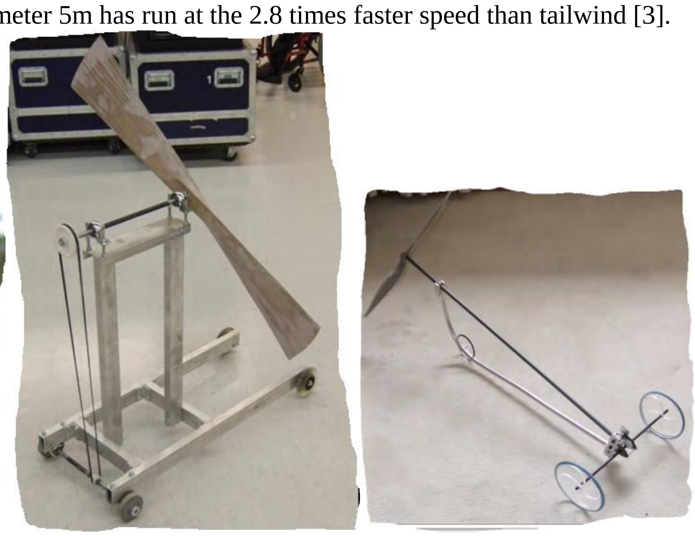


Figure 2: Cavallaro's model [5]

Blackbird wind cart. Photo: Steve Morris

Figure 3: Blackbird [3]

On the other hand, the theoretical verification was given by Mark Drela [7]. The absorbed energy through rotation of wheel or water turbine is transferred to air propeller, and the thrust is generated by the air propeller. If the running resistance of the car or ship is sufficiently smaller than the thrust, the car or the ship can advance faster than the tailwind speed. The key of the theory is to consider the phenomena relative to the sea or the ship and to focus on the energy absorption by the wheel or the water turbine and the thrust generation by the air propeller.

Rick Cavallaro's experiment using a treadmill [5] has verified experimentally that a car can advance faster than the tailwind utilizing the wind energy alone. When a car runs at the same speed as the tailwind, the relative velocity becomes zero. This corresponds to the car on the belt of the treadmill moving to the rear. To common sense, the car seems likely to be swept away to the rear.

In this experiment, the rotation of the wheel is transferred to the air propeller, and the air propeller generates the thrust, and the car moves to forward. The car tries to move forward, even if the treadmill tilted uphill. Hence, DDFTTW proposed by Jack Goodman was verified completely in the laboratory.

When a car start from rest as in case of Goodman's experiment, the tailwind pushes the car and make move the car forward by rotating wheel using the torque generated at the propeller. As the car velocity increases, the effective attack angle of the air flowing to the propeller foil becomes small, and the lift force produced at the foil generates the propeller thrust. The rotation of the propeller is transferred from the wheel. Namely, the thrust is generated by the pushing force and torque acting on the propeller. When the car velocity exceeds a certain limit, the relationship is reversed. The wheel absorbs the energy and transformed into propulsive energy by the propeller.

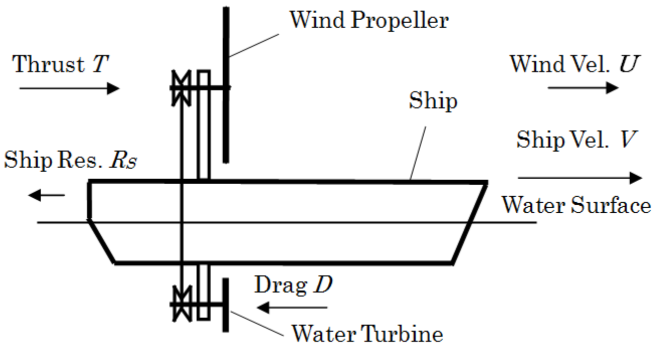
If we change the viewpoint and replace "a car with a wheel" by "a ship with a water turbine", a similar phenomenon could occur. The ship could run faster than the tailwind, if the resistance of a ship is very small. If we consider a sea current instead of a wind, it becomes same as the positioning of a floating platform in the sea current.

We suppose that a carriage is disposed on the rail of the water circulating tank. If an air propeller is placed on the carriage, and the water turbine is hanged from the carriage, then, the situation is very similar to those described above.

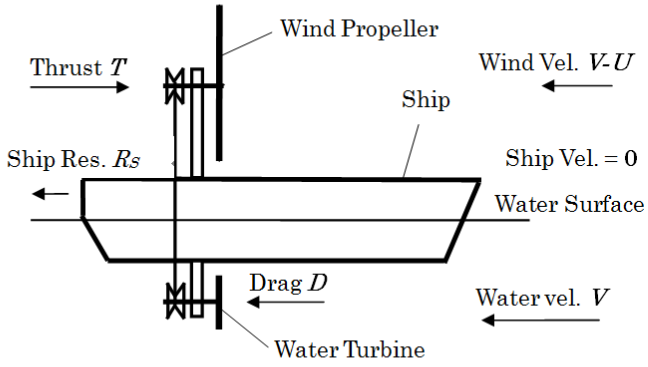
The research results on DDWFTTW by Goodman, Cavallaro and Drela are very valuable. At the same time, the research on a wind powered ship using wind turbine by Neil Bose [9] should also be remembered.

2. FUNDAMENTAL PROBLEM

As shown in Figure 4, a wind powered ship is going forward at a speed faster than a tailwind. Figures 4(a) and 4(b) are drawn relative to the sea and ship, respectively. Since the ship velocity V is bigger than the tailwind velocity U, the wind relative to the ship becomes a headwind with velocity $V-U$ as shown in Figure 4(b).



(a) Relative to sea

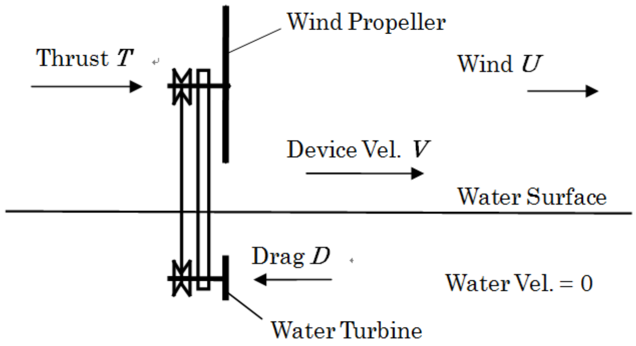


(b) Relative to ship

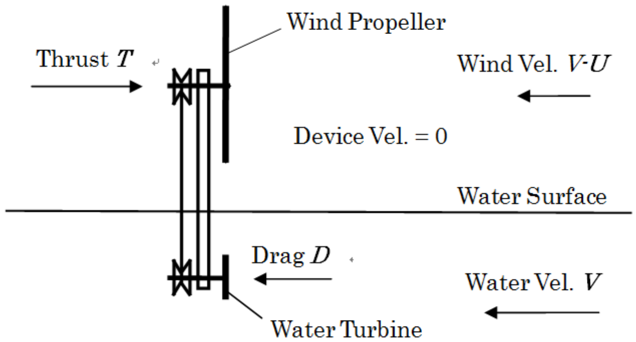
Figure 4: Tailwind (*V*>*U*)

The ship is in a stream with velocity V. We discuss the possibility of extracting energy from a water turbine and producing thrust by an air propeller.

In order to study the possibility, we consider a device as shown in Figure 5 and discuss the possibility if the thrust T produced by the air propeller becomes bigger than the drag D. We investigate the condition for $T > U$.



(a) Relative to sea



(b) Relative to device

Figure 5: Fundamental device in Tailwind (*V*>*U*)

Let P_{Wat} and P_{Air} refer to the power extracted by the water turbine and the power consumed at the air propeller, respectively. The drag D_{Wat} and thrust T_{Air} are acting on the water turbine and the air propeller, respectively. If the net thrust is denoted by T_{Net}, we have

$$T_{Net} = T_{Air} - D_{Wat}, \tag{1}$$

$$P_{Air} = P_{Wat}. \tag{2}$$

Applying the momentum theory in Appendices A and B, we have

$$D_{Wat} = 2\rho_{Wat}(V - u_{Wat})u_{Wat}A_{Wat}, \quad P_{Wat} = 2\rho_{Wat}(V - u_{Wat})u_{Wat}{}^2 A_{Wat}, \tag{3}$$

$$T_{Air} = 2\rho_{Air}(u_{Air} - (V - U))u_{Air}A_{Air}\,, \quad P_{Air} = 2\rho_{Air}(u_{Air} - (V - U))u_{Air}{}^2 A_{Air}\,, \tag{4}$$

where ρ_{Air} and ρ_{Wat} are density of air and water, respectively. A_{Air} and A_{Wat} refer to the actuator disk area of the air propeller and that of the water turbine, respectively. The velocity on the actuator disk of the air propeller and that of the water turbine are referred to u_{Air} and u_{Wat}, respectively.

Substituting Eqs. (3) and (4) into Eq. (1), we have

$$T_{Net} = 2\rho_{Air}A_{Air}\left(u_{Air} - (V - U)\right)u_{Air} - 2\rho_{Wat}A_{Wat}(V - u_{Wat})u_{Wat}\,. \tag{5}$$

Rewriting Eq. (5), we obtain

$$C_{TNet} = \frac{T_{Net}}{\frac{1}{2}\rho_{Wat}A_{Wat}V^2} = 4\frac{\rho_{Air}}{\rho_{Wat}}\frac{A_{Air}}{A_{Wat}}\left(\frac{u_{Air}}{V} - \left(1 - \frac{U}{V}\right)\right)\frac{u_{Air}}{V} - 4\left(1 - \frac{u_{Wat}}{V}\right)\frac{u_{Wat}}{V}\,, \tag{6}$$

where C_{TNet} the thrust coefficient of the net thrust.

Substituting Eqs. (3) and (4) into Eq. (2), we have

$$2\rho_{Air}(u_{Air} - (V - U))u_{Air}{}^2 A_{Air} = 2\rho_{Wat}(V - u_{Wat})u_{Wat}{}^2 A_{Wat}\,. \tag{7}$$

Rewriting Eq. (7), we obtain

$$\left(\frac{u_{Air}}{V}\right)^3 - \frac{V - U}{V}\left(\frac{u_{Air}}{V}\right)^2 - \frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{V}\right)\left(\frac{u_{Wat}}{V}\right)^2\frac{A_{Wat}}{A_{Air}} = 0\,. \tag{8}$$

When the ship velocity V is equal to the wind velocity U :

$$V = U\,, \tag{9}$$

Eqs. (6) and (8) become

$$C_{TNet} = 4\frac{\rho_{Air}}{\rho_{Wat}}\frac{A_{Air}}{A_{Wat}}\left[\frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{V}\right)\left(\frac{u_{Wat}}{V}\right)^2\frac{A_{Wat}}{A_{Air}}\right]^{2/3} - 4\left(1 - \frac{u_{Wat}}{V}\right)\frac{u_{Wat}}{V}\,, \tag{10}$$

$$\frac{u_{Air}}{V} = \left[\frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{V}\right)\left(\frac{u_{Wat}}{V}\right)^2\frac{A_{Wat}}{A_{Air}}\right]^{1/3}. \tag{11}$$

These conditions are identical to the conditions of the position keeping of a floating body in a current with velocity V [8].

The net thrust coefficient C_{TNet} becomes non-negative when

$$\frac{A_{Air}}{A_{Wat}} \geq \frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{V}\right)\left(\frac{u_{Wat}}{V}\right)^{-1}. \tag{12}$$

On the other hand, since the condition that the efficiency of the energy absorbing turbine becomes the maximum of 60% is given by (→ Appendix A)

$$\frac{u_{Wat}}{V} = \frac{2}{3}\,, \tag{13}$$

we obtain from Eq. (12)

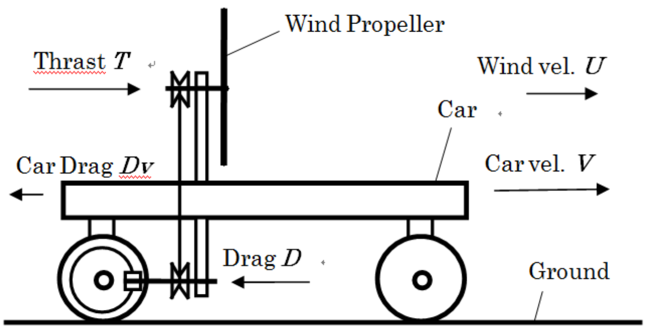
$$\frac{A_{Air}}{A_{Wat}} \geq 800 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^{-1} = 400\,. \tag{14}$$

This value is pretty big.

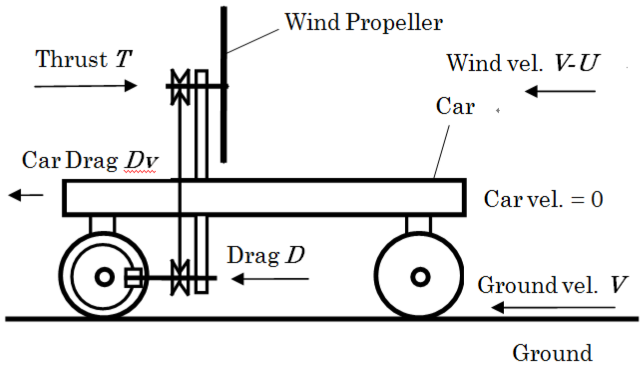
In case of the wind powered car, the energy is extracted from the rotation of the wheel. Hence, the efficiency is nearly 100%. Furthermore, the ground resistance of a car is much smaller than the water resistance of a ship. This means that the wind powered car can drive using the wind energy alone.

3. A MECHANICAL MODEL OF A WIND POWERED CAR RUNNING AT SUPER WIND VELOCITY IN TAILWIND [7]

Figure 6 illustrates a wind powered car in tailwind. Figures 6(a) and 6(b) are the illustration relative to the ground and that relative to the car, respectively. In the tailwind, the car extracts the energy through the rotation of the wheel and generates the thrust by the air propeller. We study the possibility whether the car can go forward at a speed faster than the tailwind.



(a) Relative to ground



(b) Relative to car

Figure 6: Wind powered car in Tailwind (V>U)

The thrust T_{Air} produced by the air propeller and the drag D_{Gro} acting on the wheel must satisfy the equilibrium of the forces, and the extracted energy P_{Gro} and the consumed power P_{Air} follow the balance of the powers:

$$T_{Air} - D_{Gro} = R_{Gro} + R_{Air}\,, \tag{15}$$

$$P_{Gro} + P_{Add} = P_{Air} + P_{Loss}\,, \tag{16}$$

where R_{Gro} and R_{Air} refer to the ground resistance of the wheels other than the energy extracting ones and the air resistance of the car, respectively. P_{Add} and P_{Loss} are the additional power and the loss power, respectively.

The wheel drag D_{Gro} and the propeller thrust T_{Air} are given by

$$D_{Gro} = \kappa W\,, \tag{17}$$

$$P_{Gro} = D_{Gro}V = \kappa WV\,. \tag{18}$$

where W and κ are the total weight of the car and the frictional coefficient, respectively.

Applying the momentum theory of a propeller in Appendix B, we have

$$T_{Air} = 2\rho_{Air}(u_{Air} - (V-U))u_{Air}A_{Air}\,, \tag{19a}$$

$$P_{Air} = 2\rho_{Air}(u_{Air} - (V-U))u_{Air}{}^{2}A_{Air} = A_{Air}u_{Air}\,, \tag{19b}$$

where ρ_{Air}, A_{Air} and u_{Air} refer to the density of air, the actuator disk area of the air propeller and the velocity of air on the actuator disk, respectively.

The ground resistance R_{Gro} and the air resistance R_{Air} are given by

$$R_{Gro} = \mu W\,, \tag{20}$$

$$R_{Air} = \frac{1}{2}\rho_{Air}C_{DAir}(V-U)^{2}S_{Air}\,, \tag{21}$$

where μ is the sum of the fiction coefficients of wheels except the energy absorbing wheels, and C_{DAir} is the air drag coefficient of the car.

Substituting Eqs. (19)-(21) into Eqs. (15) and (16), we obtain

$$2\rho_{Air}A_{Air}\left(u_{Air}-(V-U)\right)u_{Air}-\kappa W=\mu W+\frac{1}{2}\rho_{Air}C_{DAir}(V-U)^2 S_{Air}\,, \tag{22}$$

$$-2\rho_{Air}(u_{Air}-(V-U))u_{Air}{}^2 A_{Air}+\kappa WV=-P_{Add}+P_{Loss}\,. \tag{23}$$

Rewriting Eq. (22), we have

$$\left(\frac{u_{Air}}{V}-\left(1-\frac{U}{V}\right)\right)\frac{u_{Air}}{V}=\frac{\kappa_{Car}W_{Car}}{2\rho_{Air}A_{Air}V^2}+\frac{\mu_{Car}W_{Car}}{2\rho_{Air}A_{Air}V^2}+\frac{1}{4}C_{DAir}\left(1-\frac{U}{V}\right)^2\frac{S_{Air}}{A_{Air}}\,. \tag{24}$$

Since Eq. (24) is a second order algebraic equation with respect to the unknown u_{Air}/V , we obtain solving the equation

$$\frac{u_{Air}}{V}=\frac{1}{2}\left(1-\frac{U}{V}\right)+\sqrt{\frac{1}{4}\left(1-\frac{U}{V}\right)^2+\frac{\kappa_{Car}W_{Car}}{2\rho_{Air}A_{Air}V^2}+\frac{\mu_{Car}W_{Car}}{2\rho_{Air}A_{Air}V^2}+\frac{1}{4}C_{DAir}\left(1-\frac{U}{V}\right)^2\frac{S_{Air}}{A_{Air}}}\,. \tag{25}$$

Rewriting Eq. (23), we have

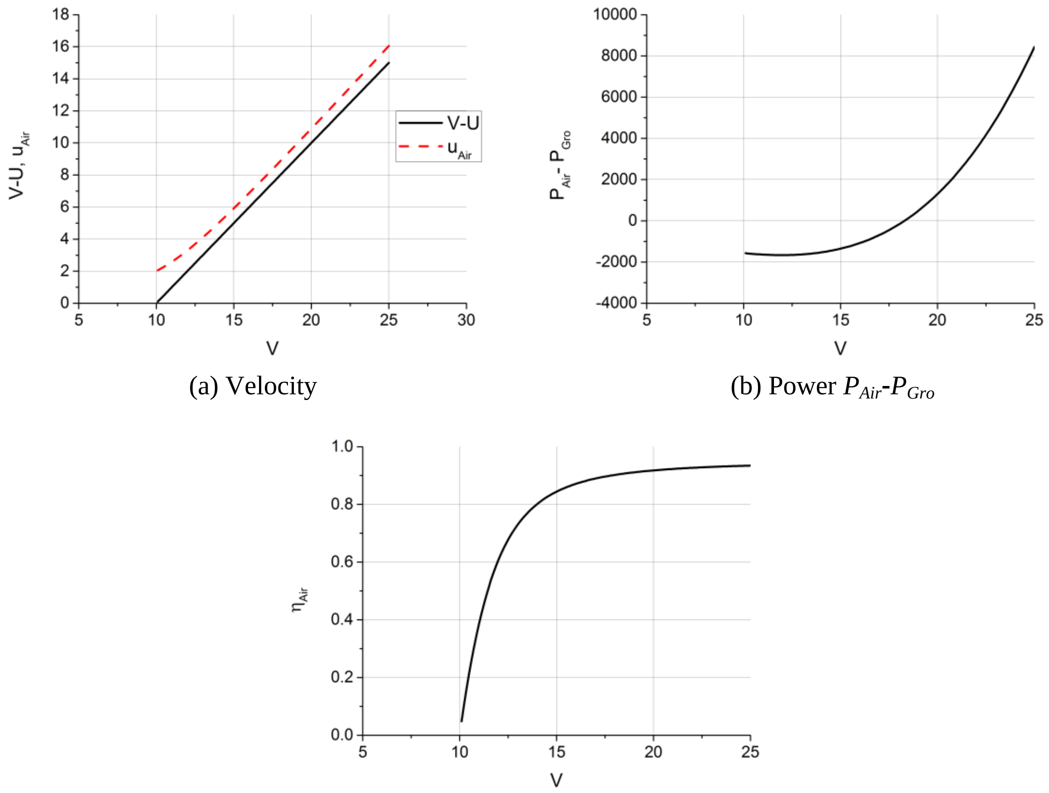
$$\frac{-P_{Add}+P_{Loss}}{2\rho_{Air}A_{Air}V^3}=-\left(\frac{u_{Air}}{V}-\left(1-\frac{U}{V}\right)\right)\left(\frac{u_{Air}}{V}\right)^2+\frac{\kappa_{Car}W_{Car}}{2\rho_{Air}A_{Air}U^2}\left(\frac{U}{V}\right)^2. \tag{26}$$

The numerical results using Eqs. (25) and (26) are shown below. We used the following calculation parameters:

$$A_{Air}=\pi\times 2.5^2=19.63m^2\,,\ S_{Air}=2.0\times 0.5+0.5\times 7m^2=4.5m^2\,,\ M=200\,kg\,,$$

$$\rho_{Air}=1.25\,,\ \kappa_{Car}=0.1\,,\ \mu_{Car}=0.001\,,\ C_{DAir}=1.0\,,\ U=10.0ms^{-1}\,. \tag{30}$$

The calculation results are shown in Figure 7, where the velocity V of the car with respect to the ground is changed from $10ms^{-1}$ to $25ms^{-1}$ under $U=10ms^{-1}$. As shown in Figure 7(b), $P_{Air}-P_{Gro}$ is negative in $V=10\sim 18.3ms^{-1}$. Namely, since the power P_{Gro} absorbed by the wheel is bigger than the power P_{Air} consumed at the air propeller, the car can go forward within the velocity range using wind energy alone, if the power loss P_{Loss} is zero. At $V=18.3ms^{-1}$, the additional power P_{Add} is equal to zero. We call the velocity the self-running velocity.



(a) Velocity

(b) Power P_{Air}-P_{Gro}

(c) Efficiency

Figure 7: Running performance of wind powered car (C_{DAir}=1.0, S_{Air}=4.5m^2)

We now obtain the velocity where the wind powered car can go forward using only wind energy. We assume the additional power P_{Add} and P_{Loss} are equal to zero. We have from Eq. (26)

$$-\left(\frac{u_{Air}}{V}-\left(1-\frac{U}{V}\right)\right)\left(\frac{u_{Air}}{V}\right)^2+\frac{\kappa_{Car}W_{Car}}{2\rho_{Air}A_{Air}U^2}\left(\frac{U}{V}\right)^2=0\,. \tag{27}$$

Now, Equations (24) and (27) are algebraic equations with unknowns U/V and u_{Air}/V. We solve the problem using Newton-Raphson method.

Firstly, differentiating Eq. (25), we obtain

$$d\left(\frac{u_{Air}}{V}\right)=\left[-\frac{1}{2}+\frac{-\frac{1}{4}\left(1-\frac{U}{V}\right)+\frac{\kappa_{Car}W_{Car}}{2\rho_{Air}A_{Air}U^2}\left(\frac{U}{V}\right)+\frac{\mu_{Car}W_{Car}}{2\rho_{Air}A_{Air}U^2}\left(\frac{U}{V}\right)-\frac{1}{4}C_{DAir}\left(1-\frac{U}{V}\right)\frac{S_{Air}}{A_{Air}}}{\sqrt{\frac{1}{4}\left(1-\frac{U}{V}\right)^2+\frac{\kappa_{Car}W_{Car}}{2\rho_{Air}A_{Air}V^2}+\frac{\mu_{Car}W_{Car}}{2\rho_{Air}A_{Air}V^2}+\frac{1}{4}C_{DAir}\left(1-\frac{U}{V}\right)^2\frac{S_{Air}}{A_{Air}}}}\right]d\left(\frac{U}{V}\right). \tag{28}$$

On the other hand, from the first order approximation of Eq. (27), we derive

$$\begin{aligned}&-\left(\frac{u_{Air}}{V}-\left(1-\frac{U}{V}\right)\right)\left(\frac{u_{Air}}{V}\right)^2+\frac{\kappa_{Car}W_{Car}}{2\rho_{Air}A_{Air}U^2}\left(\frac{U}{V}\right)^2\\&-\left(3\frac{u_{Air}}{V}-2\left(1-\frac{U}{V}\right)\right)\frac{u_{Air}}{V}d\left(\frac{u_{Air}}{V}\right)+\left(-\left(\frac{u_{Air}}{V}\right)^2+\frac{\kappa_{Car}W_{Car}}{\rho_{Air}A_{Air}U^2}\frac{U}{V}\right)d\left(\frac{U}{V}\right)=0\end{aligned}\,. \tag{29}$$

Substituting Eq. (28) into Eq. (29), we obtain $d(U/V)$. Using this, we renew the old value $[U/V]_{Old}$ to the new value $[U/V]_{New}$:

$$[U/V]_{New}=[U/V]_{Old}+d(U/V). \tag{30}$$

We iterate the process until the convergence is obtained.

The following numerical examples are based on Blackbird experiment by Cavallaro [3]. We show two examples. In the first example, we assume a big air drag coefficient. In the second example, a small air drag coefficient is assumed.

In the first example for the bigger drag coefficient C_{DAir}, the calculation parameters are set as follows:

$$A_{Air}=\pi\times 2.5^2=19.63m^2,\ S_{Air}=2.0\times 0.5+0.5\times 7m^2=4.5m^2,\ M=200\,kg,$$
$$\rho_{Air}=1.25,\ \kappa_{Car}=0.1,\ \mu_{Car}=0.001,\ C_{DAir}=1.0,\ U=1.6\sim 25.0ms^{-1}. \tag{31}$$

The start value of U/V in the iteration is taken as

$$U/V=0.5, \tag{32}$$

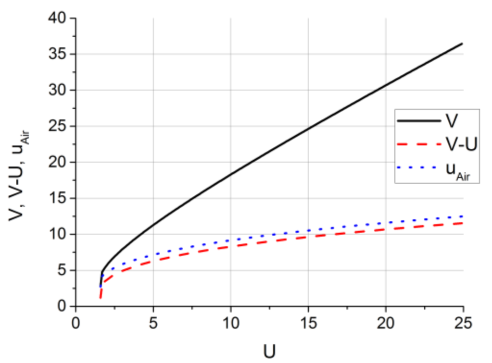
and we used 40 for the number of the iteration. The calculation results are shown in Figure 8.

In the second example for the smaller drag coefficient C_{DAir}, the calculation parameters are set as follows:

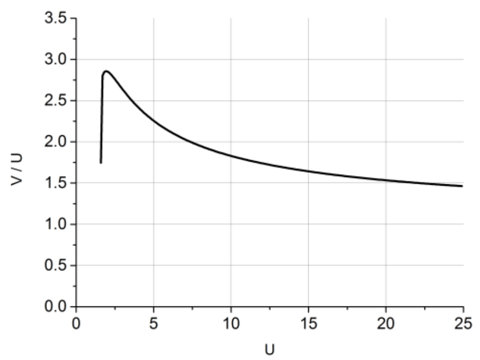
$$A_{Air}=\pi\times 2.5^2=19.63m^2,\ S_{Air}=2.0\times 0.5+0.5\times 5m^2=3.5m^2,\ M=200\,kg,$$
$$\rho_{Air}=1.25,\ \kappa_{Car}=0.1,\ \mu_{Car}=0.001,\ C_{DAir}=0.05,\ U=1.6\sim 25.0ms^{-1}. \tag{33}$$

We used the same starting value given by Eq. (32) and the same iteration number as in the first example. The calculation results are shown in Figure 9.

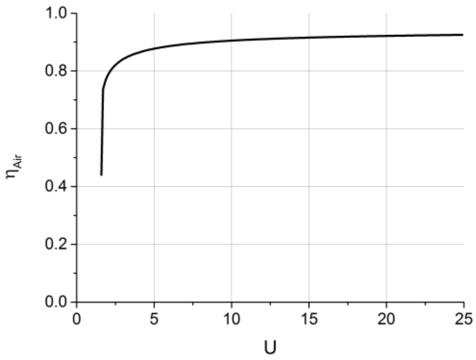
The calculation results in Figures 8 and 9 tell us that the wind powered car can realize the super-wind-velocity running. In the experiment using Blackbird, the maximum of V/U was 2.8. The result in Figure 9(b) supports this, if we make the air drag small using a streamlining of the car shape.



(a) Velocity

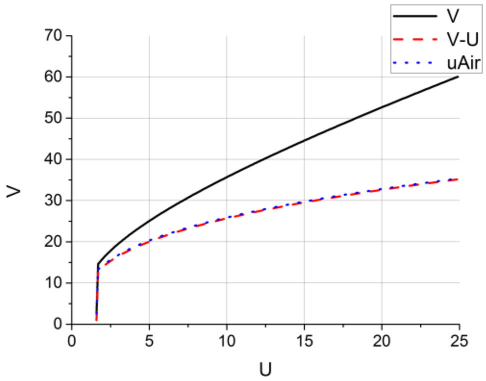


(b) Ratio of car velocity to wind velocity

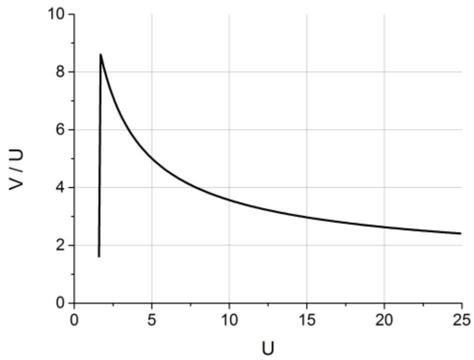


(c) Efficiency

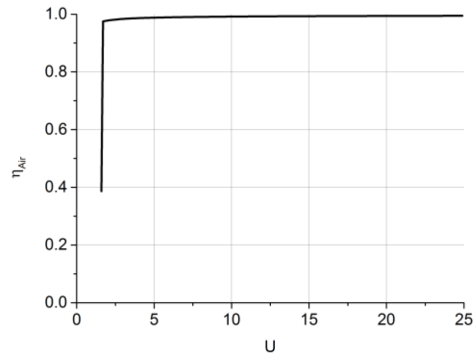
Figure 8: Self-running performance of wind powered car (C_{DAir}=1.0, S_{Air}=4.5m^2)



(a) Velocity



(b) Ratio of car velocity to wind velocity



(c) Efficiency

Figure 9: Self-running performance of wind powered car (C_{DAir}=0.05, S_{Air}=3.5m^2)

4. A MECHANICAL MODEL OF A WIND POWERED SHIP ADVANCING AT SUPER WIND VELOCITY IN TAILWIND [7]

Figure 4 illustrates a wind powered ship in tailwind. Figures 4(a) and 4(b) are the illustration relative to the water and that relative to the ship, respectively. In the tailwind, the ship extracts the energy through the water turbine and generates the thrust by the air propeller. We study the possibility whether the ship can go forward at a speed faster than the tailwind.

The thrust T_{Air} produced by the air propeller and the drag D_{Wat} acting on the water turbine must satisfy the equilibrium of the forces, and the extracted energy P_{Wat} and the consumed power P_{Air} follow the balance of the powers:

$$T_{Air} - D_{Wat} = R_{Wat} + R_{Air}\,, \qquad P_{Wat} + P_{Add} = P_{Air} + P_{Loss}\,, \qquad (34,\ 35)$$

where R_{Wat} and R_{Air} refer to the water and air resistances of the ship, respectively. P_{Add} and P_{Loss} are the additional power and the loss power, respectively.

Applying the momentum theory of an energy absorbing turbine in Appendix A, we have

$$a_{Wat} = 1 - \frac{u_{Wat}}{V} \;\dots\; 0 \le a_{Wat} \le 1, \quad D_{Wat} = 2\rho_{Wat}(V - u_{Wat})u_{Wat}A_{Wat}, \quad P_{Wat} = 2\rho_{Wat}(V - u_{Wat})u_{Wat}{}^2 A_{Wat} = D_{Wat}u_{Wat}, \tag{36a, b, c}$$

where a_{Wat} and ρ_{Wat} are the axial induction factor and the density of water, respectively. A_{Wat} refers to the actuator disk area of the water turbine, respectively. The velocity on the actuator disk of the water turbine is referred to u_{Wat}. In the following, we assume the maximum efficiency of the water turbine:

$$a_{Wat} = \frac{1}{3} \text{ or } \frac{u_{Wat}}{V} = \frac{2}{3}. \tag{37}$$

Applying the momentum theory of a propeller in Appendix B, we have

$$a_{Air} = \frac{u_{Air}}{(V-U)} - 1 \;\dots\; 0 \le a_{Air}, \quad T_{Air} = 2\rho_{Air}(u_{Air} - (V-U))u_{Air}A_{Air}, \quad P_{Air} = 2\rho_{Air}(u_{Air} - (V-U))u_{Air}{}^2 A_{Air} = T_{Air}u_{Air} \tag{38a, b, c}$$

where a_{Air}, ρ_{Air}, A_{Air} and u_{Air} refer to the axial induction factor, the density of air, the actuator disk area of the air propeller and the velocity of air on the actuator disk, respectively.

The water resistance R_{Wat} and the air resistance R_{Air} acting on the ship are given by

$$R_{Wat} = \frac{1}{2}\rho_{Wat}C_{DWat}V^2\Delta_{Wat}{}^{2/3} \qquad R_{Air} = \frac{1}{2}\rho_{Air}C_{DAir}(V-U)^2 S_{Air} \tag{39, 40}$$

where C_{DWat} and C_{DAir} are the water and air drag coefficients of the ship. Δ_{DWat} is the displacement of the ship, and S_{Air} is the area above the sea surface projected on the plane normal to the direction of ship advance.

Substituting Eqs. (36), (38), (39) and (40) into Eqs. (34) and (35), we have

$$2\rho_{Air}(u_{Air} - (V-U))u_{Air}A_{Air} - 2\rho_{Wat}(V - u_{Wat})u_{Wat}A_{Wat} = \frac{1}{2}\rho_{Wat}C_{DWat}V^2\Delta_{Wat}{}^{2/3} + \frac{1}{2}\rho_{Air}C_{DAir}(V-U)^2 S_{Air}, \tag{41}$$

$$2\rho_{Wat}(V - u_{Wat})u_{Wat}{}^2 A_{Wat} - 2\rho_{Air}(u_{Air} - (V-U))u_{Air}{}^2 A_{Air} = -P_{Add} + P_{Loss}. \tag{42}$$

Rewriting Eq. (41), we obtain

$$\left(\frac{u_{Air}}{V} - \left(1 - \frac{U}{V}\right)\right)\frac{u_{Air}}{V} - \frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{V}\right)\frac{u_{Wat}}{V}\frac{A_{Wat}}{A_{Air}} = \frac{1}{4}\frac{\rho_{Wat}}{\rho_{Air}}C_{DWat}\frac{\Delta_{Wat}{}^{2/3}}{A_{Air}} + \frac{1}{4}C_{DAir}\left(1 - \frac{U}{V}\right)^2\frac{S_{Air}}{A_{Air}}. \tag{43}$$

Since Eq. (43) is a second order algebraic equation with respect to the unknown u_{Air}/V, we obtain solving the equation

$$\frac{u_{Air}}{V} = \frac{1}{2}\left(1 - \frac{U}{V}\right) + \sqrt{\frac{1}{4}\left(1 - \frac{U}{V}\right)^2 + \frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{V}\right)\frac{u_{Wat}}{V}\frac{A_{Wat}}{A_{Air}} + \frac{1}{4}\frac{\rho_{Wat}}{\rho_{Air}}C_{DWat}\frac{\Delta_{Wat}{}^{2/3}}{A_{Air}} + \frac{1}{4}C_{DAir}\left(1 - \frac{U}{V}\right)^2\frac{S_{Air}}{A_{Air}}}. \tag{44}$$

Rewriting Eq. (42), we have

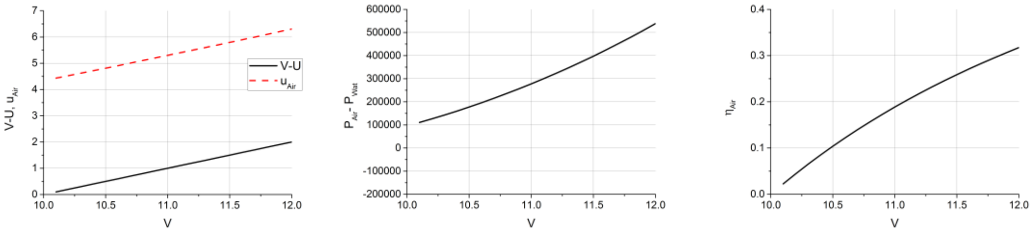
$$\frac{-P_{Add} + P_{Loss}}{2\rho_{Air}V^3 A_{Air}} = \frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{V}\right)\left(\frac{u_{Wat}}{V}\right)^2\frac{A_{Wat}}{A_{Air}} - \left(\frac{u_{Air}}{V} - \left(1 - \frac{U}{V}\right)\right)\left(\frac{u_{Air}}{V}\right)^2. \tag{45}$$

In case of a ship, the maximum efficiency of the energy absorbing water turbine is 60%, and the water resistance of a ship is big in comparison with a car. This implies that the size of the air propeller becomes unrealistically big, if we try to propel a ship using the wind power alone. We demonstrate this by a numerical example.

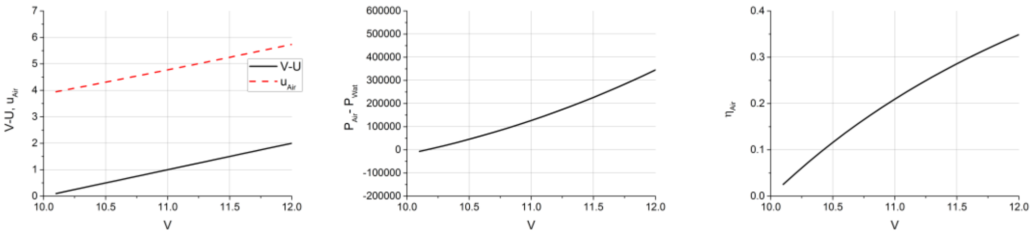
We used the following calculation parameters:

$$\begin{gathered} L = 80m,\ B = 11.61m,\ d = 5.144\,m,\ C_b = 0.5658\,m,\ \Delta = C_b LBd = 2703m^3, \\ R_{Wat} = 1m,\ R_{Air} = 40 \sim 50m,\ R_{Air}/R_{Wat} = 40, 45, 50,\ S_{Air} = 8 \times 3m^2 = 24m^2, \\ C_{DWat} = 0.01,\ C_{DAir} = 0.1,\ \rho_{Wat} = 1000,\ \rho_{Air} = 1.25,\ \rho_{Wat}/\rho_{Air} = 800, \\ U = 10ms^{-1},\ V = 10 \sim 12ms^{-1}. \end{gathered} \tag{46}$$

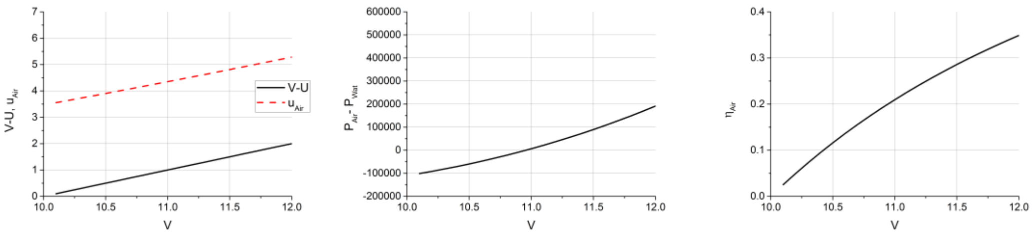
The calculation results are shown in Figure 10. As can be understood from Eq. (35), when $P_{Air} - P_{Wat}$ is negative, the power P_{Wat} absorbed by the water turbine is bigger than the power P_{Air} consumed at the air propeller. Namely, the ship can go forward using the wind power alone. When the radius of the air propeller R_{Air} is 45 times of the radius R_{Wat} of the water turbine, the condition is satisfied in the neighborhood of $V = 10\,m/s$. When R_{Air} is 50 times of R_{Wat}, the condition is satisfied if $V < 10.9\,m/s$. This implies that if R_{Air} is increased, $u_{Air}/(V-U)$ is decreased and the propulsive efficiency is increased.



(a) R_{Wat}=1m, R_{Air}=40m



(b) R_{Wat}=1m, R_{Air}=45m



(c) R_{Wat}=1m, R_{Air}=50m

Figure 10: Condition of self-running.

When R_{Air} is 80 times of R_{Wat}, the condition is satisfied when $V < 13.7\,m/s$ as shown in Figure 11. R_{Air}/R_{Wat} is extremely big. According to Eq. (14), R_{Air}/R_{Wat} should be bigger than 20. Hence, there is a limit for the effort making R_{Air}/R_{Wat} small.

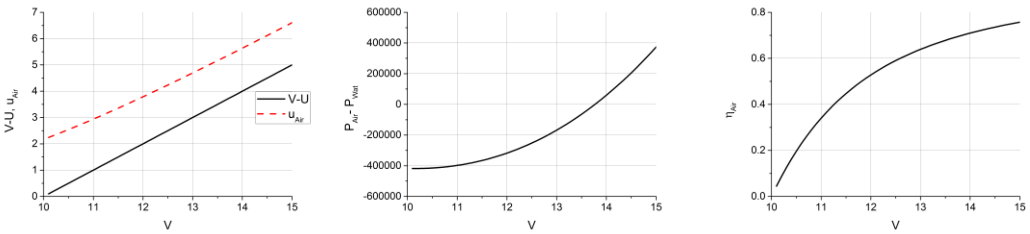


Figure 11: Self-running using very large air propeller
(R_{Wat}=1m, R_{Air}=80 m, V=10~15ms^{-1}).

Anyway, it is not realistic to drive a mono-hull ship using wind power alone. A twin-hull ship or tri-hull ship might give a possibility.

Furthermore, if the ship hull becomes huge, the possibility increases. For example, if L, B and d becomes 10 times bigger in Eq. (46), that is, $L = 800m$, $B = 116.1m$ and $d = 51.44m$, the displacement Δ becomes 1000 times bigger, but the wetted surface area of the ship becomes only 100 times bigger. Hence, the water resistance coefficient C_{DWat} becomes 1/10 or 0.001. If we assume $R_{Wat} = 10m$ and $R_{Air} = 400\,m$, the ship go forward in $V < 17.6\,m/s$ at $U = 10ms^{-1}$ using wind power alone.

If we make a twin hull ship using two mono-hulls mentioned above and use one water turbine and one air propeller, the radius R_{Wat} becomes $14.1m$ and radius R_{Air} becomes $566m$. We could set the distance between the mono-hulls to $600m$.

If we use the wind propelling device as an auxiliary power for propulsion, the possibility would be much more realistic. We add a following condition to one given by Eq. (46):

$$P_{Add} = \frac{1}{2}\rho_{Wat}U^2 C_{DWat}\Delta^{2/3} \times U \Big/ 0.6 = 1{,}666{,}000\,kNms^{-1}\,. \tag{47}$$

Figure 10(a) is enlarged in Figure 12. From Figure 12, the ship velocity V satisfying $P_{Air} - P_{Wat} = P_{Add}$ is $14.6\,m/s$. We could expect a big energy saving effect. We consider to build a twin-hull ship using two mono-hulls specified by Eq. (46). If we use one water turbine and one air propeller, the radius R_{Wat} of water turbine becomes $1.41m$, and the radius R_{Air} of water turbine becomes $56.6m$. We could set the distance between the mono-hulls to 60m. An image of the twin-hull ship is shown in Figure 13.

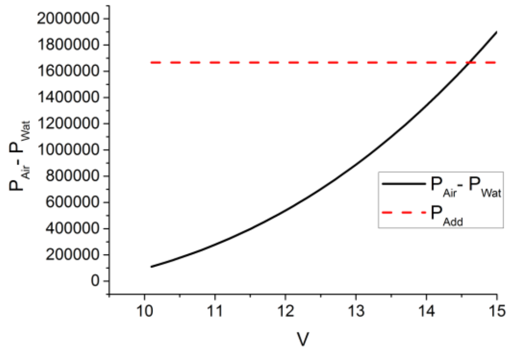


Figure 12: Energy saving or velocity increase using wind power

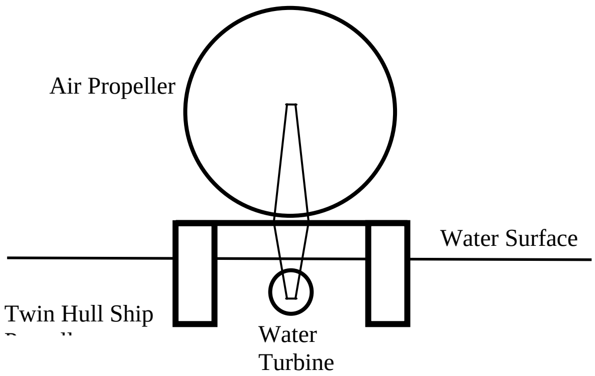


Figure 13: Twin hull ship

In the following, we consider that we use the wind propelling device as an auxiliary power for propulsion. We assume the additional power P_{Add} :

$$P_{Add} = \frac{1}{2} \rho_{Wat} V_0^{\,2} C_{DWat} \Delta^{2/3} \times V_0 \Big/ 0.6 \tag{48}$$

and the loss power equal to zero, namely

$$\frac{-P_{Add}}{2\rho_{Air} V^3 A_{Air}} = \frac{\rho_{Wat}}{\rho_{Air}} \left(1 - \frac{u_{Wat}}{V}\right) \left(\frac{u_{Wat}}{V}\right)^2 \frac{A_{Wat}}{A_{Air}} - \left(\frac{u_{Air}}{V} - \left(1 - \frac{U}{V}\right)\right) \left(\frac{u_{Air}}{V}\right)^2 . \tag{49}$$

In Eqs. (43) and (49), U/V, u_{Air}/V and u_{Wat}/V are unknowns. However, assuming the maximum energy extraction from water current, u_{Wat}/V is assumed as

$$\frac{u_{Wat}}{V} = \frac{2}{3} . \tag{50}$$

Figure 12 shows ship velocity satisfying Eqs. (43) and (49). We obtain the velocity using Newton-Raphson method. We obtain from Eq. (43)

$$\frac{d\left(\frac{u_{Air}}{V}\right)}{d\left(\frac{U}{V}\right)} = -\frac{1}{2} + \frac{-\frac{1}{4}\left(1 - \frac{U}{V}\right) + \frac{1}{2}\frac{\rho_{Wat}}{\rho_{Air}}\left(1 - 2\frac{u_{Wat}}{U}\frac{U}{V}\right)\frac{u_{Wat}}{U}\frac{A_{Wat}}{A_{Air}} - \frac{1}{4} C_{DAir}\left(1 - \frac{U}{V}\right)\frac{S_{Air}}{A_{Air}}}{\sqrt{\frac{1}{4}\left(1 - \frac{U}{V}\right)^2 + \frac{\rho_{Wat}}{\rho_{Air}}\left(1 - \frac{u_{Wat}}{U}\frac{U}{V}\right)\frac{u_{Wat}}{U}\frac{U}{V}\frac{A_{Wat}}{A_{Air}} + \frac{1}{4}\frac{\rho_{Wat}}{\rho_{Air}} C_{DWat}\frac{\Delta_{Wat}^{2/3}}{A_{Air}} + \frac{1}{4} C_{DAir}\left(1 - \frac{U}{V}\right)^2 \frac{S_{Air}}{A_{Air}}}} . \tag{51}$$

On the other hand, we derive the first approximation of Eq. (49)

$$\left[\left(\left(\frac{u_{Air}}{V}\right)^2-\frac{3P_{Add}}{2\rho_{Air}U^3A_{Air}}\left(\frac{U}{V}\right)^2-2\frac{\rho_{Wat}}{\rho_{Air}}\left(\frac{u_{Wat}}{U}\right)^2\left(\frac{U}{V}\right)\frac{A_{Wat}}{A_{Air}}+3\frac{\rho_{Wat}}{\rho_{Air}}\left(\frac{u_{Wat}}{U}\right)^3\left(\frac{U}{V}\right)^2\frac{A_{Wat}}{A_{Air}}\right)\right.$$
$$\left.+\left(3\left(\frac{u_{Air}}{V}\right)^2-2\left(1-\frac{U}{V}\right)\frac{u_{Air}}{V}\right)\left(d\left(\frac{u_{Air}}{V}\right)\Big/d\left(\frac{U}{V}\right)\right)\right]d\left(\frac{U}{V}\right) \qquad (52)$$
$$=\frac{P_{Add}}{2\rho_{Air}V^3A_{Air}}+\frac{\rho_{Wat}}{\rho_{Air}}\left(1-\frac{u_{Wat}}{V}\right)\left(\frac{u_{Wat}}{V}\right)^2\frac{A_{Wat}}{A_{Air}}-\left(\frac{u_{Air}}{V}-\left(1-\frac{U}{V}\right)\right)\left(\frac{u_{Air}}{V}\right)^2$$

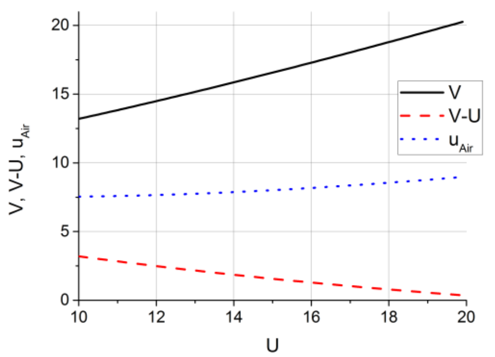
Substituting Eq. (51) into Eq. (52), we obtain $d(U/V)$. Using this, we renew the old value $[U/V]_{Old}$ to the new value $[U/V]_{New}$:

$$[U/V]_{New}=[U/V]_{Old}+d(U/V). \qquad (53)$$

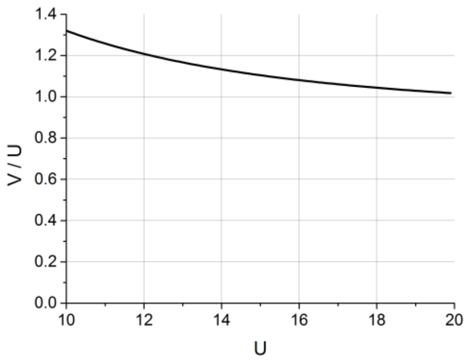
We iterate the process until the convergence is obtained.

The calculation parameters are given below:

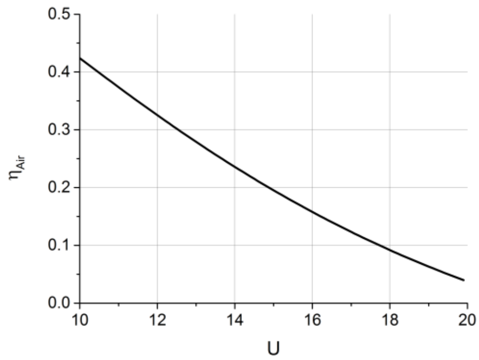
$$L=80m,\ B=11.61m,\ d=5.144\,m,\ C_b=0.5658\,m,\ \Delta=C_bLBd=2703m^3,$$
$$R_{Wat}=1m,\ R_{Air}=40m,\ R_{Air}/R_{Wat}=40,\ S_{Air}=8\times 3m^2=24m^2,$$
$$C_{DWat}=0.01,\ C_{DAir}=0.1,\ \rho_{Wat}=1000,\ \rho_{Air}=1.25,\ \rho_{Wat}/\rho_{Air}=800,$$
$$U=10\sim 20ms^{-1},\ V_0=10ms^{-1}\text{ or }16.17\times 10^6kNm^2s^{-1} \qquad (54)$$



(a) Velocity



(b) Ratio of car velocity to wind velocity



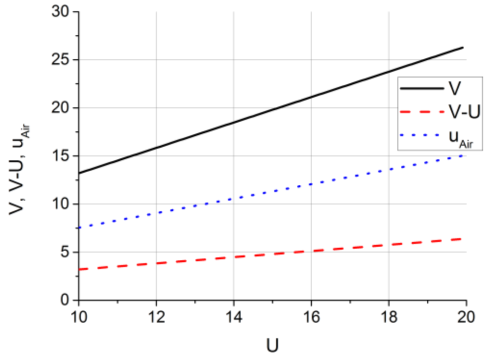
(c) Efficiency

Figure 14: Running performance of wind powered ship with additive power (C_{DWat}=0.01, R_{Air}/R_{Wat}=40, V_0=10ms^{-1})

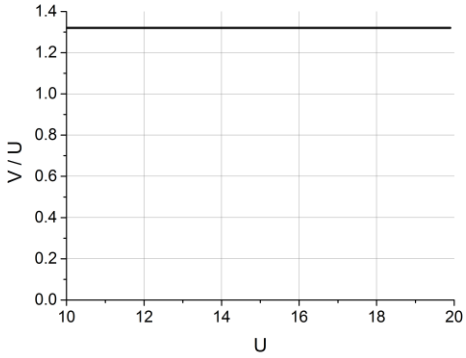
The start value of U/V in the iteration is taken as

$$U/V=0.5, \qquad (55)$$

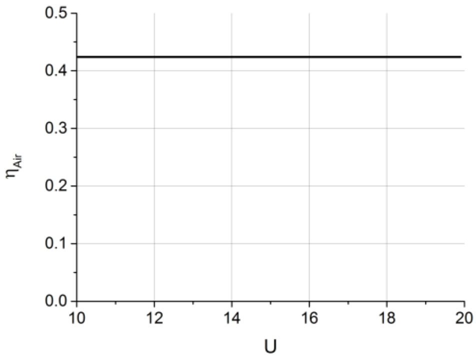
and we used 40 for the number of the iteration. The calculation results are shown in Figure 14. For example, $V=13.2ms^{-1}$ is obtained at $U=10ms^{-1}$. A very big energy saving is attained.



(a) Velocity



(b) Ratio of car velocity to wind velocity



(c) Efficiency

Figure 15: Running performance of wind powered ship with additive power (C_{DWat}=0.01, R_{Air}/R_{Wat}=40)

Next, we consider a case where the wind velocity U is set constant. The additional power P_{Add} is given by

$$P_{Add} = \frac{1}{2}\rho_{Wat}U^2 C_{DWat}\Delta^{2/3} \times U \Big/ 0.6 \,. \tag{56}$$

The calculation parameters are same as the previous example, and the same conditions are used for the iteration calculation. The calculation results are shown in Figure 15. Interesting results are obtained from the viewpoint of energy saving.

5. CONCLUSIONS

The theoretical verification of the principle of DDWFTTW was given by Mark Drela [7]. The absorbed energy through rotation of wheel or water turbine is transferred to air propeller, and the thrust is generated by the air propeller. He showed that the car or the ship can advance faster than the tailwind speed, if the running resistance of the car or ship is sufficiently smaller than the thrust. However, Drela's theory is not sufficient to obtain the design guideline.

In the present paper, Drela's theory was extended further, and the context of the practical problems was clarified. Namely, the size of a car or a ship was specified, and the relationship between the running velocity of the car or the ship and that of the tailwind was clarified. As the result, the experimental result that the wind powered car called Blackbird can run at the 2.8 times faster velocity than the tailwind [3] was verified.

Furthermore, we clarified the problems we would face when we apply the present idea to the wind powered vessel. Since the wind powered car absorbs the energy from the wheel rotating on the ground, the efficiency of energy absorption is high, and the running resistance is small. On the other hand, the wind powered ship extracts the energy from the water current using the water turbine. Hence, the efficiency of the energy absorption is low and the propulsive resistance is large. Since the ratio of air density to water density is very small, the diameter of the air propeller becomes at least 20 times bigger than that of the water turbine. Furthermore, in order to overcome the propulsive resistance of the ship, we must raise the efficiency of the air propeller as much as possible. Hence, we must make the diameter of the air propeller very big. This means we can't apply the present idea to a mono hull ship. We are obliged to apply it to a twin hull ship. Therefore, we proposed to apply it to a twin hull ship and use the wind energy as an auxiliary energy. In that case, we can expect a big energy saving effect.

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APPENDIX A. MOMENTUM THEORY OF AN ENERGY HARVESTING TURBINE [10,11]

In the following, we summarize Betz's momentum theory on a windmill. Betz obtained a limit of the theoretical efficiency called Betz limit. He treated the rotor of the windmill as an actuator disk as shown in Figure A1. He applied the conservation laws of mass, momentum and energy to obtain his theory.

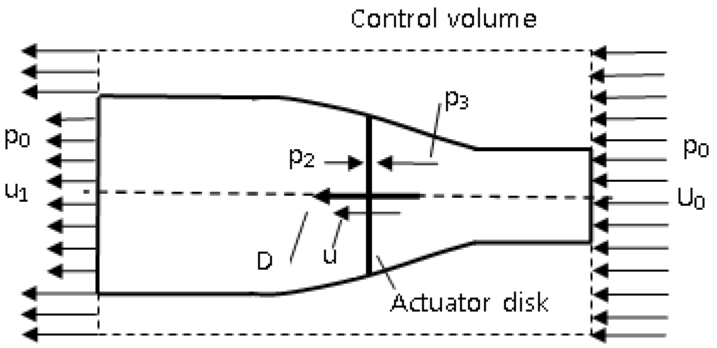

Figure A1: Energy harvesting turbine

From the continuity law, we have

$$U_0 A_0 = u_1 A_1 = uA, \tag{A1}$$

where A_0, A and A_1 are the areas of the upstream inlet, actuator disk and downstream outlet, respectively. U_0, u and u_1 are the axial velocities in A_0, A and A_1, respectively. The pressures in A_0, A^-, A^+ and A_1 are denoted by p_0, p_2, p_3 and p_1, respectively. A^- and A^+ refer to the suction and pressure sides of the actuator disk, respectively. D is a drag force acting on the actuator disk.

Applying the conservation of momentum, we have

$$D = \rho\left(U_0^2 A_0 - u_1^2 A_1\right) = \rho u_1 A_1 \left(U_0 - u_1\right), \tag{A2}$$

where ρ is the density of fluid.

Conservation of Energy or Bernoulli's theorem gives

$$\frac{1}{2}\rho U_0^2 + p_0 = \frac{1}{2}\rho u^2 + p_3, \qquad \frac{1}{2}\rho u^2 + p_2 = \frac{1}{2}\rho u_1^2 + p_0. \tag{A3a, b}$$

Adding Eqs. (A3a) and (A3b), we obtain

$$p_3 - p_2 = \frac{1}{2}\rho\left(U_0^2 - u_1^2\right). \tag{A4}$$

Then, we have

$$D = A(p_3 - p_2) = \frac{1}{2}\rho A(U_0^{\ 2} - u_1^{\ 2}). \tag{A5}$$

From Eqs. (A1), (A2) and (A5), we obtain

$$u = \frac{1}{2}(U_0 + u_1). \tag{A6}$$

Introducing the axial induction coefficient a :

$$a = \frac{U_0 - u}{U_0} = 1 - \frac{u}{U_0} \quad \text{or} \quad u = (1-a)U_0 , \tag{A7}$$

we obtain from Eqs. (A6) and (A7)

$$u_1 = (1-2a)U_0 . \tag{A8}$$

Substituting Eq. (A8) into Eq. (A5), the drag D is given by

$$D = \frac{1}{2}\rho A(U_0^{\ 2} - u_1^{\ 2}) = \frac{1}{2}\rho U_0^{\ 2} A\left(1 - \left(\frac{u_1}{U_0}\right)^2\right) = \frac{1}{2}\rho U_0^{\ 2} A\left(1-(1-2a)^2\right) = 2\rho U_0^{\ 2} Aa(1-a) = 2\rho A(U_0 - u)u . \tag{A9}$$

The power P extracted from the flow is obtained as

$$P = \frac{1}{2}\rho\left(A_0 U_0^{\ 3} - A_1 u_1^{\ 3}\right) = \frac{1}{2}\rho\left(U_0^{\ 2} - u_1^{\ 2}\right)uA = \frac{1}{2}\rho U_0^{\ 3}\left(1 - \frac{u_1}{U_0}\right)\left(1 + \frac{u_1}{U_0}\right)\frac{u}{U_0}A$$

$$= \frac{1}{2}\rho U_0^{\ 3}(1-(1-2a))(1+(1-2a))(1-a)A = 2\rho U_0^{\ 3} a(1-a)^2 A = 2\rho(U_0 - u)u^2 A . \tag{A10}$$

Then, the efficiency C_P becomes

$$C_P \equiv \frac{P}{\frac{1}{2}\rho U_0^{\ 3} A} = 4a(1-a)^2 . \tag{A11}$$

From Eq. (A11), the maximum efficiency is obtained at

$$a_{\max} = 1/3 , \tag{A12}$$

With

$$C_{P\max} = \frac{16}{27} = 0.593 . \tag{A13}$$

APPENDIX B. MOMENTUM THEORY OF A THRUST PRODUCING TURBINE [12,13]

In the following, the momentum theory of a thrust producing turbine is summarized. Almost same notations as in Appendix A are used except the axial force acting on the actuator. We use T for the thrust produced by the actuator disk.

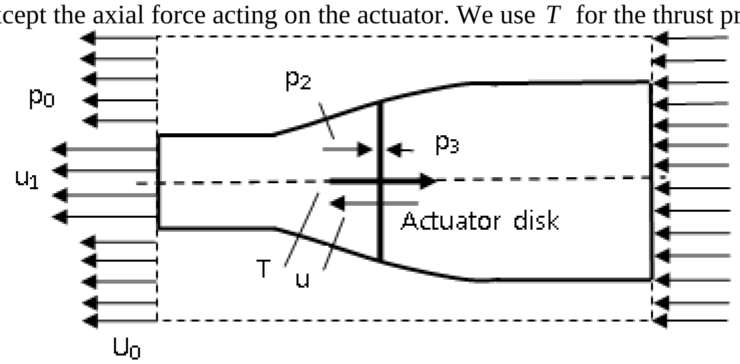

Figure B1: Propeller

From the continuity law, we have

$$U_0 A_0 = u_1 A_1 = uA . \tag{B1}$$

Applying the conservation of momentum, we have

$$T = \rho\left(u_1^{\ 2} A_1 - U_0^{\ 2} A_0\right) = \rho u_1 A_1 (u_1 - U_0). \tag{B2}$$

Conservation of Energy or Bernoulli's theorem gives

$$\frac{1}{2}\rho U_0^{\ 2} + p_0 = \frac{1}{2}\rho u^2 + p_3 , \qquad \frac{1}{2}\rho u^2 + p_2 = \frac{1}{2}\rho u_1^{\ 2} + p_0 . \tag{B3a, b}$$

Adding Eqs. (B3a) and (B3b), we have

$$p_2 - p_3 = \frac{1}{2}\rho\left(u_1^{\ 2} - U_0^{\ 2}\right). \tag{B4}$$

Then, the thrust T is given by

$$T = A(p_2 - p_3) = \frac{1}{2}\rho A\left(u_1^{\ 2} - U_0^{\ 2}\right). \tag{B5}$$

From Eqs. (B1), (B2) and (B5), we have

$$u = \frac{1}{2}(U_0 + u_1). \tag{B6}$$

Introducing the axial induction coefficient a :

$$a = \frac{u - U_0}{U_0} = \frac{u}{U_0} - 1 \text{ or } u = (1+a)U_0, \tag{B7}$$

we obtain from Eqs. (B6) and (B7)

$$u_1 = (1+2a)U_0. \tag{B8}$$

Substituting Eq. (B8) into Eq. (B5), the thrust T is given by

$$T = \frac{1}{2}\rho A\left(u_1^{\ 2} - U_0^{\ 2}\right) = \frac{1}{2}\rho U_0^{\ 2} A\left(\left(\frac{u_1}{U_0}\right)^2 - 1\right) = \frac{1}{2}\rho U_0^{\ 2} A\left((1+2a)^2 - 1\right) = 2\rho U_0^{\ 2} Aa(1+a) = 2\rho A(u - U_0)u. \tag{B9}$$

The power P given to from the flow is obtained as

$$\begin{aligned} P &= \frac{1}{2}\rho\left(A_1 u_1^{\ 3} - A_0 U_0^{\ 3}\right) = \frac{1}{2}\rho\left(u_1^{\ 2} - U_0^{\ 2}\right)uA = \frac{1}{2}\rho U_0^{\ 3}\left(\frac{u_1}{U_0} - 1\right)\left(\frac{u_1}{U_0} + 1\right)\frac{u}{U_0}A \\ &= \frac{1}{2}\rho U_0^{\ 3}\left((1+2a) - 1\right)\left((1+2a)+1\right)(1+a)A = 2\rho U_0^{\ 3} a(1+a)^2 A = 2\rho(u - U_0)u^2 A. \end{aligned} \tag{B10}$$

Then, the efficiency C_P becomes

$$C_P \equiv \frac{U_0 T}{P} = \frac{2\rho U_0^{\ 3} Aa(1+a)}{2\rho U_0^{\ 3} a(1+a)^2 A} = \frac{1}{1+a} = \frac{U_0}{u} = \frac{2U_0}{U_0 + u_1}. \tag{B11}$$

The maximum efficiency $C_{P\max} = 1$ is obtained at $a = 0$ with $T = 0$, but $C_{P\max} \approx 0.8$ actually.

At the bollard condition or $U_0 = 0$, the theory is modified slightly. Namely, the continuity equation becomes

$$u_1 A_1 = uA. \tag{B12}$$

The thrust T from the momentum conservation is given by

$$T = \rho A_1 u_1^{\ 2}. \tag{B13}$$

The energy conservation or Bernoulli's theorem becomes

$$p_0 = \frac{1}{2}\rho u^2 + p_3, \qquad \frac{1}{2}\rho u^2 + p_2 = \frac{1}{2}\rho u_1^{\ 2} + p_0. \tag{B14a, b}$$

Hence, we obtain

$$p_2 - p_3 = \frac{1}{2}\rho u_1^{\ 2}. \tag{B15}$$

Then, the thrust T is obtained as

$$T = A(p_2 - p_3) = \frac{1}{2}\rho A u_1^{\ 2}. \tag{B16}$$

From Eqs. (B12), (B13) and (B16), we have

$$A_1 = \frac{1}{2}A \text{ or } u = \frac{1}{2}u_1. \tag{B17}$$

Now, the power P given to the flow is given by

$$P = \frac{1}{2}\rho A_1 u_1^{\ 3} = \frac{1}{4}\rho A u_1^{\ 3}. \tag{B18}$$