The Solutions and Periods of Some Considered Non-Linear Difference Equation Systems

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ABSTRACT— ***We find the periods of some non-linear discrete equation systems in this study. Then we obtain the solutions of these systems releted to initial values.***

Keywords— The period of discrete system, high order non-linear discrete systems

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1. INTRODUCTION

Difference equations system arise in many branches of mathematics as well as other sciences. It is a fascinating issue because it is used to solve some problems that appear to be very complicated in applied areas. It has a wide application area especially in biology, physics, social and engineering fields [1-9]. Some are following. Clark and Kulenovic examined the global stability properties and asymptotic behavior of recursive equation systems at [2]. Nasri M and at all, introduced a deterministic model for HIV infection in the presence of combination therapy with differential equations system [4]. In [1,3], Cınar C., Yalçınkaya I. and Iricanin B., Stevic S. have considered some system of nonlinear difference equations of higher order with periodic solutions. Kose H., Uslu K. and Taskara N., examined the solution dynamics of the nonlinear iteration system in [5]. In [5,7,8,9], some nonlinear discrete systems have been investigated and the dynamics of these systems have been investigated.

In this study, we consider following non-linear systems

$$p_{n+1}=\frac{1}{s_n-r_n-q_n}+\frac{1}{q_{n-1}-r_{n-1}},\qquad q_{n+1}=\frac{1}{s_n-r_n-q_n}+\frac{r_n(s_{n-1}-r_{n-1}-q_{n-1})}{r_{n-1}},$$

$$r_{n+1}=\frac{1}{s_n-r_n-q_n},\ \ s_{n+1}=\frac{2}{s_n-r_n-q_n}+\frac{r_n(s_{n-1}-r_{n-1}-q_{n-1})}{r_{n-1}}+\frac{1}{p_n-r_n},(n\geq 0)\tag{1.1}$$

with initial values $p_0,q_{-1},q_0,r_{-1},r_0,s_{-1},s_0\in\mathbb{R}-\{0\},p_0\neq r_0,s_i\neq q_i+r_i,q_i\neq r_i,i\in\{-1,0\}$,

and

$$p_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{1}{q_{n-2}-r_{n-2}},\qquad q_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_n(s_{n-2}-r_{n-2}-q_{n-2})}{r_{n-1}},$$

$$r_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}},\ \ s_{n+1}=\frac{2}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_n(s_{n-2}-r_{n-2}-q_{n-2})}{r_{n-1}}+\frac{1}{p_{n-1}-r_{n-1}},(n\geq 0)\tag{1.2}$$

with initial values $p_{-1},p_0,q_{-2},q_{-1},q_0,r_{-2},r_{-1},r_0,s_{-2},s_{-1},s_0\in\mathbb{R}-\{0\},p_i\neq r_i,i\in\{-1,0\},\ s_i\neq q_i+r_i,q_i\neq r_i,i\in\{-2,-1,0\}$.

Firstly, we give basic definitions. Let I_1, I_2, I_3 and I_4 be some intervals of real numbers and let $F_1{:}I_2\times I_3\times I_4\to I_1$, $F_2{:}I_2\times I_3\times I_4\to I_2$, $F_3{:}I_2\times I_3\times I_4\to I_3$ and $F_4{:}I_1\times I_2\times I_3\times I_4\to I_4$ be four continuously differentiable functions. It is obvious that the system of difference equations (1.3)

$$p_{n+1}=F_1(q_n,r_n,s_n),\qquad q_{n+1}=F_2(q_n,r_n,s_n)$$

$$r_{n+1}=F_3(q_n,r_n,s_n),\qquad s_{n+1}=F_4(p_n,q_n,r_n,s_n)\tag{1.3}$$

has a unique solution $\{p_n, q_n, r_n, s_n\}$ for every initial condition $(p_i, q_i, r_i, s_i) \in I_1 \times I_2 \times I_3 \times I_4$.

A solution $\{p_n, q_n, r_n, s_n\}$ of the system of difference equations (1.3) is periodic if there exist a positive integer m such that

$$p_{n+m} = p_n, \quad q_{n+m} = q_n, \quad r_{n+m} = r_n, \quad s_{n+m} = s_n$$

the smallest such positive integer m is called the prime period of the solution of difference equation system (1.3) [2,4].

2. THE SOLUTIONS AND PERIODS OF SOME CONSİDERED NON-LINEAR DIFFERENCE EQUATION SYSTEMS

The following theorems show us the period of solutions of the systems (1.1) and (1.2).

Theorem 2.1. Suppose that $\{p_n, q_n, r_n, s_n\}$ are the solutions of the difference equation system (1.1) with initial values $p_0, q_{-1}, q_0, r_{-1}, r_0, s_{-1}, s_0 \in \mathbb{R} - \{0\}, p_0 \neq r_0, s_i \neq q_i + r_i, q_i \neq r_i, i \in \{-1,0\}$. Then all solutions of the system (1.1) are periodic with period 6.

Proof: From the system (1.1), it is obtained the following equalities

$$p_{n+1} = \frac{1}{s_n - r_n - q_n} + \frac{1}{q_{n-1} - r_{n-1}}, \qquad q_{n+1} = \frac{1}{s_n - r_n - q_n} + \frac{r_n(s_{n-1} - r_{n-1} - q_{n-1})}{r_{n-1}}$$

$$r_{n+1} = \frac{1}{s_n - r_n - q_n}, \qquad s_{n+1} = \frac{2}{s_n - r_n - q_n} + \frac{r_n(s_{n-1} - r_{n-1} - q_{n-1})}{r_{n-1}} + \frac{1}{p_n - r_n},$$

$$p_{n+2} = p_n - r_n + \frac{1}{q_n - r_n}, \qquad q_{n+2} = p_n - r_n + \frac{1}{r_n}$$

$$r_{n+2} = p_n - r_n, \qquad s_{n+2} = 2(p_n - r_{n)} + \frac{1}{r_n} + q_{n-1} - r_{n-1},$$

$$p_{n+3} = \frac{1}{q_{n-1} - r_{n-1}} + \frac{r_{n-1}}{r_n(s_{n-1} - r_{n-1} - q_{n-1})}, \qquad q_{n+3} = \frac{1}{q_{n-1} - r_{n-1}} + s_n - r_n - q_n$$

$$r_{n+3} = \frac{1}{q_{n-1} - r_{n-1}}, \qquad s_{n+3} = \frac{2}{q_{n-1} - r_{n-1}} + s_n - r_n - q_n + q_n - r_n,$$

$$p_{n+4} = \frac{1}{q_n - r_n} + r_n, \qquad q_{n+4} = \frac{1}{q_n - r_n} + \frac{1}{p_n - r_n}$$

$$r_{n+4} = \frac{1}{q_n - r_n}, \qquad s_{n+4} = \frac{2}{q_n - r_n} + \frac{1}{p_n - r_n} + \frac{r_n(s_{n-1} - r_{n-1} - q_{n-1})}{r_{n-1}},$$

$$p_{n+5} = \frac{r_{n-1}}{r_n(s_{n-1} - r_{n-1} - q_{n-1})} + \frac{1}{(s_n - r_n - q_n)}, \qquad q_{n+5} = \frac{r_{n-1}}{r_n(s_{n-1} - r_{n-1} - q_{n-1})} + q_{n-1} - r_{n-1}$$

$$r_{n+5}=\frac{r_{n-1}}{r_n(s_{n-1}-r_{n-1}-q_{n-1})},\qquad s_{n+5}=\frac{2r_{n-1}}{r_n(s_{n-1}-r_{n-1}-q_{n-1})}+q_{n-1}-r_{n-1}+\frac{1}{r_n},$$

$$p_{n+6}=p_n,\qquad q_{n+6}=q_n$$
$$r_{n+6}=r_n,\qquad s_{n+6}=s_n.$$

Thus all solutions of the system (1.1) are periodic with 6 period.

Theorem 2.2. Suppose that $\{p_n,q_n,r_n,s_n\}$ are the solutions of the difference equation system (1.2) with initial values $p_{-1},p_0,q_{-2},q_{-1},q_0,r_{-2},r_{-1},r_0,s_{-2},s_{-1},s_0\in\mathbb{R}-\{0\},p_i\neq r_i,i\in\{-1,0\},\ s_i\neq q_i+r_i,q_i\neq r_i,i\in\{-2,-1,0\}$. Then all solutions of the system (1.2) are periodic with period 9.

Proof: From the system (1.2), it is obtained the following equalities

$$p_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{1}{q_{n-2}-r_{n-2}},\qquad q_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_n(s_{n-2}-r_{n-2}-q_{n-2})}{r_{n-1}},$$

$$r_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}},\qquad s_{n+1}=\frac{2}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_n(s_{n-2}-r_{n-2}-q_{n-2})}{r_{n-1}}+\frac{1}{p_{n-1}-r_{n-1}},$$

$$p_{n+2}=\frac{1}{s_n-r_n-q_n}+\frac{1}{q_{n-1}-r_{n-1}},\qquad q_{n+2}=\frac{1}{s_n-r_n-q_n}+\frac{1}{r_n},$$

$$r_{n+2}=\frac{1}{s_n-r_n-q_n},\qquad s_{n+2}=\frac{2}{s_n-r_n-q_n}+\frac{1}{r_n}+\frac{1}{p_n-r_n},$$

$$p_{n+3}=p_{n-1}-r_{n-1}+\frac{1}{q_n-r_n},\qquad q_{n+3}=p_{n-1}-r_{n-1}+s_{n-1}-r_{n-1}-q_{n-1},$$

$$r_{n+3}=p_{n-1}-r_{n-1},\qquad s_{n+3}=2(p_{n-1}-r_{n-1})+s_{n-1}-r_{n-1}-q_{n-1}+q_{n-2}-r_{n-2},$$

$$p_{n+4}=p_n-r_n+\frac{r_{n-1}}{r_n(s_{n-2}-r_{n-2}-q_{n-2})},\qquad q_{n+4}=p_n-r_n+s_n-r_n-q_n,$$

$$r_{n+4}=p_n-r_n,\qquad s_{n+4}=2(p_n-r_n)+s_n-r_n-q_n+q_{n-1}-r_{n-1},$$

$$p_{n+5}=\frac{1}{q_{n-2}-r_{n-2}}+r_n,\qquad q_{n+5}=\frac{1}{q_{n-2}-r_{n-2}}+\frac{1}{p_{n-1}-r_{n-1}},$$

$$r_{n+5}=\frac{1}{q_{n-2}-r_{n-2}},\qquad s_{n+5}=\frac{2}{q_{n-2}-r_{n-2}}+\frac{1}{p_{n-1}-r_{n-1}}+q_n-r_n,$$

$$p_{n+6}=\frac{1}{q_{n-1}-r_{n-1}}+\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}},\qquad q_{n+6}=\frac{1}{q_{n-1}-r_{n-1}}+\frac{1}{p_n-r_n},$$

$$r_{n+6}=\frac{1}{q_{n-1}-r_{n-1}},\qquad s_{n+6}=\frac{2}{q_{n-1}-r_{n-1}}+\frac{1}{p_n-r_n}+\frac{r_n(s_{n-2}-r_{n-2}-q_{n-2})}{r_{n-1}},$$

$$p_{n+7}=\frac{1}{q_n-r_n}+\frac{1}{s_n-r_n-q_n}, \qquad q_{n+7}=\frac{1}{q_n-r_n}+q_{n-2}-r_{n-2},$$

$$r_{n+7}=\frac{1}{q_n-r_n}, \qquad s_{n+7}=\frac{2}{q_n-r_n}+q_{n-2}-r_{n-2}+\frac{1}{r_n},$$

$$p_{n+8}=\frac{r_{n-1}}{r_n(s_{n-2}-r_{n-2}-q_{n-2})}+p_{n-1}-r_{n-1}, \qquad q_{n+8}=\frac{r_{n-1}}{r_n(s_{n-2}-r_{n-2}-q_{n-2})}+q_{n-1}-r_{n-1},$$

$$r_{n+8}=\frac{r_{n-1}}{r_n(s_{n-2}-r_{n-2}-q_{n-2})}, \qquad s_{n+8}=\frac{2r_{n-1}}{r_n(s_{n-2}-r_{n-2}-q_{n-2})}+q_{n-1}-r_{n-1}+s_{n-1}-r_{n-1}-q_{n-1},$$

$$p_{n+9}=p_n, \qquad q_{n+9}=q_n$$
$$r_{n+9}=r_n, \qquad s_{n+9}=s_n.$$

Thus all solutions of the system (1.1) are periodic with 9 period.

This study is related to the master's thesis of Vural Deniz.

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