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A New Bivariate Distribution with Generalized Gompertz Marginals

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ABSTRACT— ***In this paper, we introduce a new bivariate generalized Gompertz distribution, it is of Marshall-Olkin type. Some properties of the distribution are studied, as bivariate moment generating function, marginal moment generating function and conditional distribution. Parameters estimators using the maximum likelihood method are obtained. A numerical illustration is used to obtain maximum likelihood estimators (MLEs) and we study the behavior of the estimators numerically.***

Keywords— Generalized Gompertz distribution, Maximum likelihood estimators, Moment estimators, Fisher information matrix.

1. INTRODUCTION

Recently EL-Gohary et al. [3] introduced a three-parameters generalized Gompertz (GG) distribution by exponentiating the Gompertz (G) distribution as was done for the exponentiated Weibull distribution by Mudholkar et al. [5]. The exponentiation introduced an extra shape parameter in the model, which may yield more flexibility in the shape of the probability density function and hazard function. Several properties of their new distribution were established. They observed that exponential, generalized exponential and Gompertz distribution can be obtained as special cases of the GG distribution.

The main object of the paper is to introduce a new bivariate generalized Gompertz (BGG) distribution, whose marginals are GG distributions, it is obtained using a method similar to that used to obtain Marshall-Olkin bivariate exponential model Marshall and Olkin [6], Sarhan and Balakrishnan [7] and bivariate generalized exponential model of Kundu and Gupta [4]. The proposed distribution is constructed from three independent GG distributions using a maximization process. Computation of the estimators involves solving a four dimensional optimization problem. The generation of random samples from the BGG is quite straight forward, which makes it very convenient to perform the simulation experiments.

Several properties of the new distribution have been established, its joint probability density function and its joint cumulative distribution function are expressed in explicit forms. The marginals of the proposed distribution are univariate GG distribution. The joint moment generating function (MGF) of BGG distribution is obtained in explicit forms.

Suppose $\left((x_{11}, x_{21}), \dots, (x_{1n}, x_{2n})\right)$ is a random sample from the new bivariate distribution, then our aim is to compute the MLEs of the unknown parameters by solving four nonlinear equations. Monte Carlo simulations have been performed to see the behavior of the MLEs, and real data analysis has been performed for illustrative purpose.

The rest of the paper is organized as follows. In Section 2, we describe the models and discuss some different properties. Section 3 presents the joint moment generating function of proposed bivariate distribution. Section 4 obtains the parameter estimation using MLE. In section 5 we present a numerical result are obtained using real data and simulated data. Finally, a conclusion for the results is given in Section 6.

2. A NEW BIVARIATE GENERALIZED GOMPERTZ DISTRIBUTION

In this section, we briefly discuss the new BGG distribution. We start with the joint cumulative distribution function of the distribution and then derive the corresponding joint probability density function. Let X be a random variable has univariate GG distribution with parameters $\alpha, \lambda, c > 0$ has the probability density function (PDF), cumulative distribution function (CDF) respectively for > 0 ;

$$f(x:\alpha,\lambda,c) = \alpha\lambda e^{cx} e^{\frac{-\lambda}{c}(e^{cx}-1)} \left(1 - e^{\frac{-\lambda}{c}(e^{cx}-1)}\right)^{\alpha-1}, \tag{1}$$

$$F(x:\alpha,\lambda,c) = \left(1 - e^{\frac{-\lambda}{c}(e^{cx}-1)}\right)^{\alpha}. \tag{2}$$

where $\alpha > 0, \lambda > 0$ and $c > 0$ are the parameters.

2.1 The Joint Cumulative Distribution Function

Suppose that $\mathrm{U}_1 \sim GG(\alpha_1, \lambda, c)$, $\mathrm{U}_2 \sim GG(\alpha_2, \lambda, c)$ and $\mathrm{U}_3 \sim GG(\alpha_3, \lambda, c)$ and they are independently distributed. Define $\mathrm{X}_1 = \max(\mathrm{U}_1, \mathrm{U}_3)$ and $\mathrm{X}_2 = \max(\mathrm{U}_2, \mathrm{U}_3)$.Then, the bivariate vector (X_1, X_2) has a BGG distribution with parameters $\alpha_1, \alpha_2, \alpha_3, \lambda, c$.

We now study the joint distribution of the random variables X_1 and X_2 Sarhan and Balakrishnan [7] considered the following lemma of the joint CDF of the $BGG(\alpha_1, \alpha_2, \alpha_3, \lambda, c)$.

Lemma 2.1. The joint CDF of X_1 and X_2 is

$$F_{BGG}(x_1, x_2) = \left(1 - e^{\frac{-\lambda}{c}(e^{cx_1}-1)}\right)^{\alpha_1} \left(1 - e^{\frac{-\lambda}{c}(e^{cx_2}-1)}\right)^{\alpha_2} \left(1 - e^{\frac{-\lambda}{c}(e^{cz}-1)}\right)^{\alpha_3} \tag{3}$$

where $z = min(x_1, x_2)$

proof: Since

$$F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2)$$

we have

$$\begin{aligned} F(x_1, x_2) &= P(max(U_1, U_3) \le x_1, max(U_2, U_3) \le x_2) \\ &= P\big(U_1 \le x_1, U_2 \le x_2, U_3 \le min(U_2, U_3)\big) \end{aligned}$$

As $U_i (i = 1,2,3)$ are mutually independent, we readily obtain

$$\begin{aligned} F_{BGG}(x_1, x_2) &= P(U_1 \le x_1) P(U_2 \le x_2) P\big(U_3 \le min(U_2, U_3)\big) \\ &= F_{GG}(x_1; \alpha_1, \lambda, c) F_{GG}(x_2; \alpha_2, \lambda, c) F_{GG}(z; \alpha_3, \lambda, c) \end{aligned} \tag{4}$$

Substituting from (2) into (4), we obtain (3), which completes the proof of the lemma.2.1.

2.2 The Joint Probability Density Function

The following theorem gives the joint PDF of the X_1 and X_2 which is the joint PDF of $BGG(\alpha_1, \alpha_2, \alpha_3, \lambda, c)$.

Theorem 2.1. If the joint CDF of (X_1, X_2) is as in (3), the joint PDF of (X_1, X_2) is given by

$$f_{BGG}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } x_1 < x_2 \\ f_2(x_1, x_2) & \text{if } x_2 < x_1 \\ f_3(x, x) & \text{if } x_1 = x_2 = x \end{cases} \tag{5}$$

where

$$\begin{aligned} f_1(x_1, x_2) = {} & (\alpha_1 + \alpha_3) \lambda e^{cx_1} e^{\frac{-\lambda}{c}(e^{cx_1}-1)} \left(1 - e^{\frac{-\lambda}{c}(e^{cx_1}-1)}\right)^{(\alpha_1+\alpha_3)-1} \\ & \times \alpha_2 \lambda e^{cx_2} e^{\frac{-\lambda}{c}(e^{cx_2}-1)} \left(1 - e^{\frac{-\lambda}{c}(e^{cx_2}-1)}\right)^{\alpha_2 - 1} \end{aligned} \tag{6}$$

$$\begin{aligned} f_2(x_1, x_2) = {} & \alpha_1 \lambda e^{cx_1} e^{\frac{-\lambda}{c}(e^{cx_1}-1)} \left(1 - e^{\frac{-\lambda}{c}(e^{cx_1}-1)}\right)^{\alpha_1 - 1} \\ & \times (\alpha_2 + \alpha_3) \lambda e^{cx_2} e^{\frac{-\lambda}{c}(e^{cx_2}-1)} \left(1 - e^{\frac{-\lambda}{c}(e^{cx_2}-1)}\right)^{(\alpha_2+\alpha_3)-1} \end{aligned} \tag{7}$$

$$f_3(x, x) = \alpha_3 \lambda e^{cx} e^{\frac{-\lambda}{c}(e^{cx}-1)} \tag{8}$$

proof: Let us first assume that $x_1 < x_2$. Then, $f_{BGG}(x_1, x_2)$ in (3) will be denoted by $F_1(x_1, x_2)$ and becomes

$$F_1(x_1, x_2) = \left(1 - e^{\frac{-\lambda}{c}(e^{cx_1}-1)}\right)^{(\alpha_1+\alpha_3)} \left(1 - e^{\frac{-\lambda}{c}(e^{cx_2}-1)}\right)^{\alpha_2}$$

Then, upon differentiating this function w.r.t. x_1 and x_2 we obtain the expression of $f_1(x_1, x_2)$ given in (6). By the same way we obtain $f_2(x_1, x_2)$ when $x_2 < x_1$. But $f_3(x, x)$ cannot be derived in a similar way. For this reason, we use the following identity to derive $f_3(x, x)$

$$\int_0^\infty \int_0^{x_2} f_1(x_1, x_2)\, dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1, x_2)\, dx_2 dx_1 + \int_0^\infty f_3(x, x)\, dx = 1$$

Let

$$I_1 = \int_0^\infty \int_0^{x_2} f_1(x_1, x_2)\, dx_1 dx_2 \quad and \quad I_2 = \int_0^\infty \int_0^{x_1} f_2(x_1, x_2)\, dx_2 dx_1$$

Then

$$I_1 = \int_0^\infty \int_0^{x_2} (\alpha_1+\alpha_3)\lambda e^{cx_1} e^{\frac{-\lambda}{c}(e^{cx_1}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx_1}-1)}\right)^{(\alpha_1+\alpha_3)-1}$$
$$\times \alpha_2 \lambda e^{cx_2} e^{\frac{-\lambda}{c}(e^{cx_2}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx_2}-1)}\right)^{\alpha_2-1} dx_1 dx_2$$
$$= \int_0^\infty \alpha_2 \lambda e^{cx_2} e^{\frac{-\lambda}{c}(e^{cx_2}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx_2}-1)}\right)^{(\alpha_1+\alpha_2+\alpha_3)-1} dx_2 \tag{9}$$

Similarly

$$I_2 = \int_0^\infty \alpha_1 \lambda e^{cx_1} e^{\frac{-\lambda}{c}(e^{cx_1}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx_1}-1)}\right)^{(\alpha_1+\alpha_2+\alpha_3)-1} dx_1 \tag{10}$$

From (9) and (10), we then get

$$\int_0^\infty f_3(x,x)\,dx = \int_0^\infty (\alpha_1+\alpha_2+\alpha_3)\lambda e^{cx} e^{\frac{-\lambda}{c}(e^{cx}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx}-1)}\right)^{(\alpha_1+\alpha_2+\alpha_3)-1} dx$$
$$-\int_0^\infty \alpha_2 \lambda e^{cx} e^{\frac{-\lambda}{c}(e^{cx}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx}-1)}\right)^{(\alpha_1+\alpha_2+\alpha_3)-1} dx$$
$$-\int_0^\infty \alpha_1 \lambda e^{cx} e^{\frac{-\lambda}{c}(e^{cx}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx}-1)}\right)^{(\alpha_1+\alpha_2+\alpha_3)-1} dx$$

This is,

$$f_3(x,x) = \alpha_3 \lambda e^{cx} e^{\frac{-\lambda}{c}(e^{cx}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx}-1)}\right)^{(\alpha_1+\alpha_2+\alpha_3)-1} \tag{11}$$

This completes the proof of the theorem.

2.3 Marginal Probability Density Functions

The following theorem gives the marginal density function of X_1 and X_2.

Theorem 2.2. The marginal probability density functions of $X_i (i = 1,2)$ is given by

$$f_{X_i}(x_i) = (\alpha_i+\alpha_3)\lambda e^{cx_i} e^{\frac{-\lambda}{c}(e^{cx_i}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx_i}-1)}\right)^{(\alpha_i+\alpha_3)-1}$$
$$= f_{GG}(x_i; \alpha_i+\alpha_3, \lambda, c)$$

proof: The marginal cumulative distribution function of X_i , say $F(x_i)$, as follows:

$$F(x_i) = P(X_i < x_i) = P(max(U_i, U_3) \le x_i) = P(U_i \le x_i, U_3 \le x_i)$$

and since U_i is independent of U_3, we simply have

$$f_{X_i}(x_i) = \left(1-e^{\frac{-\lambda}{c}(e^{cx_i}-1)}\right)^{\alpha_i} \left(1-e^{\frac{-\lambda}{c}(e^{cx_i}-1)}\right)^{\alpha_3} = \left(1-e^{\frac{-\lambda}{c}(e^{cx_i}-1)}\right)^{\alpha_i+\alpha_3}$$
$$= F_{GG}(x_i; \alpha_i+\alpha_3, \lambda, c) \tag{12}$$

Differenting w.r.t. x_i we obtain the formula given in (12).

2.4 Conditional Probability Density Functions

Given the marginal probability density functions of X_1 and X_2 we can now derive the conditional probability density functions as presented in the following theorem.

Theorem 2.3. The conditional probability density functions of X_i, given $X_j = x_j$, $f(x_i|x_j)$, $i, j = 1,2; i \neq j$, is given by

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & \text{if } x_i < x_j, \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & \text{if } x_j < x_i, \\ f^{(3)}_{X_i|X_j}(x_i|x_j) & \text{if } x_i = x_j = x, \end{cases}$$

where

$$f^{(1)}_{X_i|X_j}(x_i|x_j) = \frac{(\alpha_1+\alpha_3)\alpha_2\lambda e^{cx_i} e^{\frac{-\lambda}{c}(e^{cx_i}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx_i}-1)}\right)^{(\alpha_1+\alpha_3)-1}}{(\alpha_2+\alpha_3)\left(1-e^{\frac{-\lambda}{c}(e^{cx_j}-1)}\right)^{\alpha_3}}$$

$$f^{(2)}_{X_i|X_j}(x_i|x_j) = \alpha_1\lambda e^{cx_i} e^{\frac{-\lambda}{c}(e^{cx_i}-1)} \left(1-e^{\frac{-\lambda}{c}(e^{cx_i}-1)}\right)^{\alpha_1-1}$$

$$f^{(3)}_{X_i|X_j}(x_i|x_j) = \frac{\alpha_3\left(1 - e^{\frac{-\lambda}{c}(e^{cx_i}-1)}\right)^{\alpha_1}}{(\alpha_2+\alpha_3)}$$

proof: The proof follows immediately by substituting the joint probability density function of (X_1, X_2) given in (6), (7) and (8) and the marginal probability density function of $X_i (i = 1,2)$ given in (12), using the relation

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_j}(x_j)}, \qquad i = 1,2.$$

3. MOMENT GENERATING FUNCTIONS

We present the joint moment generating function of (X_1, X_2) and the marginal moment generating function of $X_i(\mathrm{i} = 1,2)$.

3.1 The Marginal Moment Generating Function

We derive the marginal density functions of X_i:

Lemma 3.1. If $X_i \sim GG(\alpha_i + \alpha_3, 1,1)$, then the moment generating function of $X_i (i = 1,2)$ is given by

$$M_{X_i}(t_i) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(j+1)^k}{k!}\binom{\alpha_i+\alpha_3-1}{j}e^{(j+1)}\frac{1}{(t_i-1-k)} \tag{13}$$

proof: Consider equation (12) with $\lambda = 1$ and $c = 1$ we get

$$M_{X_i}(t_i) = (\alpha_i+\alpha_3)\int_0^{\infty} e^{-t_ix_i}\, e^{x_i}e^{-(e^{x_i}-1)}\left(1-e^{-(e^{cx_i}-1)}\right)^{(\alpha_i+\alpha_3)-1}dx_i \tag{14}$$

Since $0 < e^{-(e^{cx_i}-1)} < 1$ for $x_i > 0$, then by using the binomial series expansion given by

$$\left(1-e^{-(e^{x_i}-1)}\right)^{(\alpha_i+\alpha_3)-1} = \sum_{j=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^j\, e^{-j(e^{x_i}-1)} \tag{15}$$

Substituting from (15) into (14) we get

$$M_{X_i}(t_i) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^j\, \boldsymbol{e}^{(\boldsymbol{j}+\mathbf{1})}\int_{\mathbf{0}}^{\infty}\boldsymbol{e}^{-(\boldsymbol{t_i}-\mathbf{1})x_i}\,\boldsymbol{e}^{-e^{x_i}(\boldsymbol{j}+\mathbf{1})}\boldsymbol{d}x_i$$

using the Taylor series expansion of $\boldsymbol{e}^{-e^{x_i}(\boldsymbol{j}+\mathbf{1})}$ we get

$$\boldsymbol{e}^{-e^{x_i}(\boldsymbol{j}+\mathbf{1})} = \sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^k e^{kx_i}}{k!} \tag{16}$$

Substituting from the relation in (15) we get

$$M_{X_i}(t_i) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(j+1)^k}{k!}\binom{\alpha_i+\alpha_3-1}{j}\boldsymbol{e}^{(\boldsymbol{j}+\mathbf{1})}\int_{\mathbf{0}}^{\infty}\boldsymbol{e}^{-(\boldsymbol{t_i}-\mathbf{1}-\boldsymbol{k})x_i}\,\boldsymbol{d}x_i$$

from which we readily derive the expression of $M_{X_i}(t_i)$ given in (13).
Note that the moment generating function $M_{X_i}(t_i)$ can be used, instead of the marginal pdf $f_{X_i}(x_i)$, to derive the marginal expectation of X_i as

$$E(X_i) = -\frac{d}{dt_i}M_{X_i}(t_i)\Big|_{t_i=0}$$

From (13), we obtain

$$-\frac{d}{dt_i}M_{X_i}(t_i) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(j+1)^k}{k!}\binom{\alpha_i+\alpha_3-1}{j}\boldsymbol{e}^{(\boldsymbol{j}+\mathbf{1})}\frac{1}{(t_i-1-k)^2}$$

Similarly, the second moment of X_i can be derived from $M_{X_i}(t_i)$ as its second derivative at $t_i = 0$. The expression for the function $M_{X_i}(t_i)$ in (13) can be used to derive the rth moment of X_i as given below

$$E(X_i^r) = -\frac{d}{dt_i}M_{X_i}(t_i) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(j+1)^k}{k!}\binom{\alpha_i+\alpha_3-1}{j}\boldsymbol{e}^{(\boldsymbol{j}+\mathbf{1})}\frac{r!}{(t_i-1-k)^{r+1}}$$

3.2 The Joint Moment Generating Function

The joint moment generating function of (X_1, X_2) can be derived as follows:

Theorem 2.3. If $(X_1, X_2) \sim (\alpha_1, \alpha_2, \alpha_3, 1,1)$, then the joint moment generating function of (X_1, X_2) is given by

$$M(t_1,t_2)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_3)\alpha_2(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_1-1-j)(t_2-1-m)}e^{(i+k+2)}\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{k}$$

$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_3)\alpha_2(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_1-1-j)(t_1+t_2-2-m-j)}e^{(i+k+2)}\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{k}$$

$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_2+\alpha_3)\alpha_1(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_2-1-j)(t_1-1-m)}e^{(i+k+2)}\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{k}$$

$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_2+\alpha_3)\alpha_1(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_2-1-j)(t_1+t_2-2-m-j)}e^{(i+k+2)}\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{k}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\alpha_3(-1)^{i+j}(i+1)^j(k+1)^m}{j!\,(t_1+t_2-1-j)}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}e^{(i+1)} \tag{17}$$

proof: Sarhan and Balakrishnan [7] introduced the definition of the joint moment generating function of (X_1,X_2) as follows

$$M(t_1,t_2)=E\left(e^{-(t_1x_1+t_2x_2)}\right)=\int_0^\infty\int_0^\infty e^{-(t_1x_1+t_2x_2)}f(x_1,x_2)dx_1dx_2$$
$$M(t_1,t_2)=\int_0^\infty\int_0^{x_2} e^{-(t_1x_1+t_2x_2)}f_1(x_1,x_2)dx_1dx_2+\int_0^\infty\int_0^{x_1} e^{-(t_1x_1+t_2x_2)}f_2(x_1,x_2)dx_2dx_1$$
$$+\int_0^\infty e^{-(t_1+t_2)x}f_3(x,x)dx$$

Let

$$I_1=\int_0^\infty\int_0^{x_2} e^{-(t_1x_1+t_2x_2)}f_1(x_1,x_2)dx_1dx_2,\quad I_2=\int_0^\infty\int_0^{x_1} e^{-(t_1x_1+t_2x_2)}f_2(x_1,x_2)dx_2dx_1$$
$$I_3=\int_0^\infty e^{-(t_1+t_2)x}f_3(x,x)dx$$

Substituting from (6) into I_1 we get

$$I_1=\int_0^\infty\int_0^{x_2} e^{-(t_1x_1+t_2x_2)}(\alpha_1+\alpha_3)e^{x_1}e^{-(e^{x_1}-1)}\left(1-e^{-(e^{x_1}-1)}\right)^{(\alpha_1+\alpha_3)-1}$$
$$\times\alpha_2e^{x_2}e^{-(e^{x_2}-1)}\left(1-e^{-(e^{x_2}-1)}\right)^{\alpha_2-1}dx_1dx_2$$
$$I_1=(\alpha_1+\alpha_3)\alpha_2\int_0^\infty e^{-t_2x_2}e^{x_2}e^{-(e^{x_2}-1)}\left(1-e^{-(e^{x_2}-1)}\right)^{\alpha_2-1}$$
$$\times\int_0^{x_2} e^{-t_1x_1}e^{x_1}e^{-(e^{x_1}-1)}\left(1-e^{-(e^{x_1}-1)}\right)^{(\alpha_1+\alpha_3)-1}dx_1dx_2$$

Let

$$I_{11}=\int_0^{x_2} e^{-t_1x_1}e^{x_1}e^{-(e^{x_1}-1)}\left(1-e^{-(e^{x_1}-1)}\right)^{(\alpha_1+\alpha_3)-1}dx_1 \tag{18}$$

substituting from the relation in (15) into (18) we get

$$I_{11}=\sum_{i=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{i}(-1)^ie^{(i+1)}\int_0^{x_2}e^{-(t_1-1)x_1}e^{-(e^{x_1})(i+1)}dx_1 \tag{19}$$

substituting from the relation in (16) into (19) we get

$$I_{11}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!}\binom{\alpha_1+\alpha_3-1}{i}e^{(i+1)}\int_0^{x_2}e^{-(t_1-1-j)x_1}dx_1$$

$$I_{11} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!}\binom{\alpha_1+\alpha_3-1}{i}e^{(i+1)}\frac{\left(1-e^{-(t_1-1-j)}\right)}{(t_1-1-j)}$$

Let

$$A = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!}\binom{\alpha_1+\alpha_3-1}{i}e^{(i+1)}\frac{1}{(t_1-1-j)}$$

then

$$I_1 = (\alpha_1+\alpha_3)\alpha_2\int_0^{\infty} e^{-t_2x_2}e^{x_2}e^{-(e^{x_2}-1)}\left(1-e^{-(e^{x_2}-1)}\right)^{\alpha_2-1}\left(A-Ae^{-(t_1-1-j)x_2}\right)dx_2 \tag{20}$$

substituting from the relation in (15) into (20) we get

$$I_1 = (\alpha_1+\alpha_3)\alpha_2\sum_{k=0}^{\infty}\binom{\alpha_2-1}{k}(-1)^k e^{k+1}\int_0^{\infty} e^{-(t_2-1)x_2}e^{-(e^{x_2})(k+1)}\left(A-Ae^{-(t_1-1-j)x_2}\right)dx_2$$

using the relation in (16) I_1 becomes as following

$$I_1 = (\alpha_1+\alpha_3)\alpha_2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}(k+1)^m}{m!}\binom{\alpha_2-1}{k}e^{k+1}\int_0^{\infty} e^{-(t_2-1-m)x_2}\left(A-Ae^{-(t_1-1-j)x_2}\right)dx_2$$

Let

$$B = (\alpha_1+\alpha_3)\alpha_2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}(k+1)^m}{m!}\binom{\alpha_2-1}{k}e^{k+1}$$

then

$$I_1 = AB\int_0^{\infty} e^{-(t_2-1-m)x_2}\,dx_2 - AB\int_0^{\infty} e^{-(t_1+t_2-2-m-j)x_2}\,dx_2$$

$$\begin{aligned}I_1 = &\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_3)\alpha_2(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_1-1-j)(t_2-1-m)}e^{(i+k+2)}\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{k}\\ &-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_3)\alpha_2(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_1-1-j)(t_1+t_2-2-m-j)}e^{(i+k+2)}\binom{\alpha_1+\alpha_3-1}{i}\binom{\alpha_2-1}{k}\end{aligned}$$

Similarly we can obtain as follows

$$\begin{aligned}I_2 = &\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_2+\alpha_3)\alpha_1(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_2-1-j)(t_1-1-m)}e^{(i+k+2)}\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{k}\\ &-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha_2+\alpha_3)\alpha_1(-1)^{i+j+k+m}(i+1)^j(k+1)^m}{j!\,m!\,(t_2-1-j)(t_1+t_2-2-m-j)}e^{(i+k+2)}\binom{\alpha_2+\alpha_3-1}{i}\binom{\alpha_1-1}{k}\end{aligned}$$

And we can obtain I_3 as follows

$$I_3 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\alpha_3(-1)^{i+j}(i+1)^j(k+1)^m}{j!\,(t_1+t_2-1-j)}\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}e^{(i+1)}$$

Then we can obtain the joint moment generating function

$$M(t_1,t_2) = I_1+I_2+I_3$$

4. MAXIMUM LIKLIHOOD ESTIMATORS

Kundu and Gupta [4] used the method of maximum likelihood to estimate the unknown parameters of the bivariate generalized exponential distribution. in the same way we use the method of maximum likelihood to estimate the unknown parameters of the BGG distribution and consider c=1.

Suppose $\left((x_{11},x_{21}),\dots,(x_{1n},x_{2n})\right)$ is a random sample from $\mathrm{BGG}(\alpha_1,\alpha_2,\alpha_3,\lambda,1)$ distribution. Consider the following notation

$$n_1 = (i; X_{1i} < X_{2i}),\quad n_2 = (i; X_{1i} > X_{2i}),\quad n_3 = (i; X_{1i} = X_{2i} = X_i),\quad n = n_1+n_2+n_3$$

The likelihood of the sample of size   given by:

$$l(\alpha_1,\alpha_2,\alpha_3,\lambda)=\prod_{i=1}^{n_1} f_1(x_{1i},x_{2i})\prod_{i=1}^{n_2} f_2(x_{1i},x_{2i})\prod_{i=1}^{n_3} f_3(x_i,x_i)$$

Based on the observations, and using the density functions (6), (7) and (8) the likelihood function becomes:

$$\begin{aligned}l(\alpha_1,\alpha_2,\alpha_3,\lambda)=&\left((\alpha_1+\alpha_3)\alpha_2\lambda^2\right)^{n_1} e^{\sum_{i=1}^{n_1}x_{1i}} e^{-\lambda\sum_{i=1}^{n_1}(e^{x_{1i}}-1)} e^{\sum_{i=1}^{n_1}x_{2i}} e^{-\lambda\sum_{i=1}^{n_1}(e^{x_{2i}}-1)}\\ &\times\prod_{i=1}^{n_1}\left(1-e^{-\lambda(e^{x_{1i}}-1)}\right)^{\alpha_1+\alpha_3-1}\left(1-e^{-\lambda(e^{x_{2i}}-1)}\right)^{\alpha_2-1}\\ &\times(\alpha_1(\alpha_2+\alpha_3)\lambda^2)^{n_2} e^{\sum_{i=1}^{n_2}x_{1i}} e^{-\lambda\sum_{i=1}^{n_2}(e^{x_{1i}}-1)} e^{\sum_{i=1}^{n_2}x_{2i}} e^{-\lambda\sum_{i=1}^{n_2}(e^{x_{2i}}-1)}\\ &\times\prod_{i=1}^{n_2}\left(1-e^{-\lambda(e^{x_{1i}}-1)}\right)^{\alpha_1-1}\left(1-e^{-\lambda(e^{x_{2i}}-1)}\right)^{\alpha_2+\alpha_3-1}\\ &\times(\alpha_3\lambda)^{n_3} e^{\sum_{i=1}^{n_3}x_i} e^{-\lambda\sum_{i=1}^{n_3}(e^{x_i}-1)}\prod_{i=1}^{n_3}\left(1-e^{-\lambda(e^{x_i}-1)}\right)^{\alpha_1+\alpha_2+\alpha_3-1}\end{aligned}$$

The log-likelihood function can be written as

$$\begin{aligned}L(\alpha_1,\alpha_2,\alpha_3,\lambda)=&\,n_1 ln\left((\alpha_1+\alpha_3)\alpha_2\lambda^2\right)+\sum_{i=1}^{n_1}x_{1i}-\lambda\sum_{i=1}^{n_1}(e^{x_{1i}}-1)+\sum_{i=1}^{n_1}x_{2i}-\lambda\sum_{i=1}^{n_1}(e^{x_{2i}}-1)\\ &+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}ln\left(1-e^{-\lambda(e^{x_{1i}}-1)}\right)+(\alpha_2-1)\sum_{i=1}^{n_1}ln\left(1-e^{-\lambda(e^{x_{2i}}-1)}\right)\\ &+n_2 ln(\alpha_1(\alpha_2+\alpha_3)\lambda^2)+\sum_{i=1}^{n_2}x_{1i}-\lambda\sum_{i=1}^{n_2}(e^{x_{1i}}-1)+\sum_{i=1}^{n_2}x_{2i}-\lambda\sum_{i=1}^{n_2}(e^{x_{2i}}-1)\\ &+(\alpha_1-1)\sum_{i=1}^{n_2}ln\left(1-e^{-\lambda(e^{x_{1i}}-1)}\right)+(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}ln\left(1-e^{-\lambda(e^{x_{2i}}-1)}\right)\\ &+n_3 ln(\alpha_3\lambda)+\sum_{i=1}^{n_3}x_i-\lambda\sum_{i=1}^{n_3}(e^{x_i}-1)+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}ln\left(1-e^{-\lambda(e^{x_i}-1)}\right)\end{aligned}\qquad(21)$$

Computing the first partial derivatives of (21) with respect to α_1, α_2, α_3 and λ, and setting the results equal zeros, we get the likelihood equations as in the following form

$$\frac{\partial L}{\partial\alpha_1}=\frac{n_1}{(\alpha_1+\alpha_3)}+\sum_{i=1}^{n_1}ln\left(1-e^{-\lambda(e^{x_{1i}}-1)}\right)+\frac{n_2}{\alpha_1}+\sum_{i=1}^{n_2}ln\left(1-e^{-\lambda(e^{x_{1i}}-1)}\right)+\sum_{i=1}^{n_3}ln\left(1-e^{-\lambda(e^{x_i}-1)}\right)\qquad(22)$$

$$\frac{\partial L}{\partial\alpha_2}=\frac{n_1}{\alpha_2}+\sum_{i=1}^{n_1}ln\left(1-e^{-\lambda(e^{x_{2i}}-1)}\right)+\frac{n_2}{(\alpha_2+\alpha_3)}+\sum_{i=1}^{n_2}ln\left(1-e^{-\lambda(e^{x_{2i}}-1)}\right)+\sum_{i=1}^{n_3}ln\left(1-e^{-\lambda(e^{x_i}-1)}\right)\qquad(23)$$

$$\frac{\partial L}{\partial\alpha_3}=\frac{n_1}{(\alpha_1+\alpha_3)}+\sum_{i=1}^{n_1}ln\left(1-e^{-\lambda(e^{x_{1i}}-1)}\right)+\frac{n_2}{(\alpha_2+\alpha_3)}+\sum_{i=1}^{n_2}ln\left(1-e^{-\lambda(e^{x_{2i}}-1)}\right)+\frac{n_3}{\alpha_3}+\sum_{i=1}^{n_3}ln\left(1-e^{-\lambda(e^{x_i}-1)}\right)\qquad(24)$$

$$\begin{aligned}\frac{\partial L}{\partial\lambda}=&\frac{2n_1}{\lambda}-\sum_{i=1}^{n_1}(e^{x_{1i}}-1)-\sum_{i=1}^{n_1}(e^{x_{2i}}-1)+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\frac{(e^{x_{1i}}-1)}{\left(e^{\lambda(e^{x_{1i}}-1)}-1\right)}+(\alpha_2-1)\sum_{i=1}^{n_1}\frac{(e^{x_{2i}}-1)}{\left(e^{\lambda(e^{x_{2i}}-1)}-1\right)}\\ &+\frac{2n_2}{\lambda}-\sum_{i=1}^{n_2}(e^{x_{1i}}-1)-\sum_{i=1}^{n_2}(e^{x_{2i}}-1)+(\alpha_1-1)\sum_{i=1}^{n_2}\frac{(e^{x_{1i}}-1)}{\left(e^{\lambda(e^{x_{1i}}-1)}-1\right)}+(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}\frac{(e^{x_{2i}}-1)}{\left(e^{\lambda(e^{x_{2i}}-1)}-1\right)}\\ &+\frac{n_3}{\lambda}-\sum_{i=1}^{n_3}(e^{x_i}-1)+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}\frac{(e^{x_i}-1)}{\left(e^{-\lambda(e^{x_i}-1)}-1\right)}\end{aligned}\qquad(25)$$

To get the MLEs of the parameters α_1, α_2, α_3 and λ, we have to solve the above system of four non-linear equations with respect to α_1, α_2, α_3 and λ. The solution of equations (22), (23), (24) and (25) are not easy to solve, so numerical technique is needed to get the MLEs.

The approximate confidence intervals of the parameters based on the asymptotic distributions of their MLEs are derived. For the observed information matrix α_1, α_2, α_3 and λ, we find the second partial derivatives as follows

$$\frac{\partial^2 L}{\partial \alpha_1^2} = I_{11} = \frac{-n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{(\alpha_1)^2}, \qquad \frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2} = I_{12} = 0, \qquad \frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_3} = I_{13} = \frac{-n_1}{(\alpha_1+\alpha_3)^2}$$

$$\frac{\partial^2 L}{\partial \alpha_1 \partial \lambda} = I_{14} = \sum_{i=1}^{n_1} \frac{(e^{x_{1i}}-1)}{\left(e^{\lambda(e^{x_{1i}}-1)}-1\right)} + \sum_{i=1}^{n_2} \frac{(e^{x_{1i}}-1)}{\left(e^{\lambda(e^{x_{1i}}-1)}-1\right)} + \sum_{i=1}^{n_3} \frac{(e^{x_i}-1)}{\left(e^{\lambda(e^{x_i}-1)}-1\right)}$$

$$\frac{\partial^2 L}{\partial \alpha_2^2} = I_{22} = \frac{-n_1}{(\alpha_2)^2} - \frac{n_2}{(\alpha_2+\alpha_3)^2}, \qquad \frac{\partial^2 L}{\partial \alpha_2 \partial \alpha_3} = I_{23} = \frac{-n_2}{(\alpha_2+\alpha_3)^2}$$

$$\frac{\partial^2 L}{\partial \alpha_2 \partial \lambda} = I_{24} = \sum_{i=1}^{n_1} \frac{(e^{x_{2i}}-1)}{\left(e^{\lambda(e^{x_{2i}}-1)}-1\right)} + \sum_{i=1}^{n_2} \frac{(e^{x_{2i}}-1)}{\left(e^{\lambda(e^{x_{2i}}-1)}-1\right)} + \sum_{i=1}^{n_3} \frac{(e^{x_i}-1)}{\left(e^{-\lambda(e^{x_i}-1)}-1\right)}$$

$$\frac{\partial^2 L}{\partial \alpha_3^2} = I_{33} = \frac{-n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{(\alpha_2+\alpha_3)^2} - \frac{n_3}{(\alpha_3)^2}$$

$$\frac{\partial^2 L}{\partial \alpha_3 \partial \lambda} = I_{34} = \sum_{i=1}^{n_1} \frac{(e^{x_{1i}}-1)}{\left(e^{\lambda(e^{x_{1i}}-1)}-1\right)} + \sum_{i=1}^{n_2} \frac{(e^{x_{2i}}-1)}{\left(e^{\lambda(e^{x_{2i}}-1)}-1\right)} + \sum_{i=1}^{n_3} \frac{(e^{x_i}-1)}{\left(e^{-\lambda(e^{x_i}-1)}-1\right)}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial \lambda^2} = I_{44} = &-\frac{2n_1}{\lambda^2} - (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1} \frac{e^{\lambda(e^{x_{1i}}-1)}(e^{x_{1i}}-1)^2}{\left(e^{\lambda(e^{x_{1i}}-1)}-1\right)^2} - (\alpha_2-1)\sum_{i=1}^{n_1} \frac{e^{\lambda(e^{x_{2i}}-1)}(e^{x_{2i}}-1)^2}{\left(e^{\lambda(e^{x_{2i}}-1)}-1\right)^2} \\ &-\frac{2n_2}{\lambda^2} - (\alpha_1-1)\sum_{i=1}^{n_2} \frac{e^{\lambda(e^{x_{1i}}-1)}(e^{x_{1i}}-1)^2}{\left(e^{\lambda(e^{x_{1i}}-1)}-1\right)^2} - (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} \frac{e^{\lambda(e^{x_{2i}}-1)}(e^{x_{2i}}-1)^2}{\left(e^{\lambda(e^{x_{2i}}-1)}-1\right)^2} \\ &-\frac{n_3}{\lambda^2} - (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} \frac{e^{-\lambda(e^{x_i}-1)}(e^{x_i}-1)^2}{\left(e^{-\lambda(e^{x_i}-1)}-1\right)^2}\end{aligned}$$

Then the observed information matrix is given by

$$I = -\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}$$

so the variance-covariance matrix may be approximated ad

$$V = -\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}^{-1} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix}$$

5. SIMULATION AND DATA ANALYSIS

In this section first we present Monte Carlo simulation results to study the behavior of the MLEs and then present one data analysis results mainly for illustrative purpose.

5.1 Simulation Results

We present some simulation results to see how the MLEs behavior for different sample sizes and for different initial parameter values. We have used different sample sizes namely n = 20, 40, 60, 80 and 100 and two different sets of parameter values: Set 1: $\alpha_1 = \alpha_2 = \alpha_3 = \lambda = 1$ and Set 2: $\alpha_1 = 1.1$, $\alpha_2 = \alpha_3 = \lambda = 1$. In each case we have computed the MLEs of the unknown parameters by maximizing the log-likelihood function (20), the MLEs will be obtained using iterative procedure using Mathcad (2001) software. We compute the average estimates and mean square error over 1000 replications and the results are reported in Table 1.

Some of the points are quite clear from Table 1. In all the cases the performances of the maximum likelihood estimate are quite satisfactory. It is observed that as sample size increases the average estimates and the mean squared error decrease for all the parameters, as expected.

Table 1 The average of MLEs and the associated mean square errors (within brackets below)

	Set 1				Set 2			
Model / n	$\alpha_1=1$	$\alpha_2=1$	$\alpha_3=1$	$\lambda=1$	$\alpha_1=1.1$	$\alpha_2=1$	$\alpha_3=1$	$\lambda=1$
n=20	1.165	1.157	1.141	1.073	1.283	1.161	1.141	1.072
	(0.318)	(0.293)	(0.262)	(0.055)	(0.374)	(0.099)	(0.267)	(0.055)
n=40	1.085	1.074	1.072	1.039	1.195	1.073	1.072	1.038
	(0.126)	(0.108)	(0.089)	(0.025)	(0.146)	(0.090)	(0.093)	(0.024)
n=60	1.059	1.045	1.057	1.026	1.166	1.045	1.058	1.026
	(0.071)	(0.067)	(0.058)	(0.015)	(0.084)	(0.067)	(0.059)	(0.014)
n=80	1.038	1.045	1.03	1.019	1.144	1.045	1.029	1.019
	(0.055)	(0.05)	(0.037)	(0.011)	(0.063)	(0.051)	(0.039)	(0.011)
n=100	1.036	1.04	1.027	1.018	1.14	1.041	1.027	1.017
	(0.038)	(0.04)	(0.031)	(0.008)	(0.045)	(0.04)	(0.032)	(0.008)

5.2 Data Analysis

The following data represent the American Football (National Football League) League data and they are obtained from the matches played on three consecutive weekends in 1986. The data were first published in `Washington Post' and they are also available in Csorgo and Welsh [2].

It is a bivariate data set, and the variables X_1 and X_2 are as follows: X_1 represents the `game time' to the first points scored by kicking the ball between goal posts, and $X_2$ represents the `game time' to the first points scored by moving the ball into the end zone. These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game.

The data (scoring times in minutes and seconds) are represented in Table 2. The data set was first analyzed by Csorgo and Welsh [2], by converting the seconds to the decimal minutes, i.e. 2:03 has been converted to 2.05, 3:59 to 3.98 and so on. We also adopte the same procedure. Here also all the data points are divided by 100 just for computational purposes.

The variables X_1 and X_2 have the following structure: (i) $X_1 < X_2$ means that the first score is a field goal, (ii) $X_1 = X_2$ means the first score is a converted touchdown, (iii) $X_1 > X_2$ means the first score is an unconverted touchdown or safety. In this case the ties are exact because no `game time' elapses between a touchdown and a point-after conversion attempt. Therefore, here ties occur quite naturally and they can not be ignored. It should be noted that the possible scoring times are restricted by the duration of the game but it has been ignored similarly as in Csorgo and Welsh [2].

X_1	X_2	X_1	X_2	X_1	X_2
2.05	3.98	5.78	25.98	10.40	10.25
9.05	9.05	13.80	49.75	2.98	2.98
0.85	0.85	7.25	7.25	3.88	6.43
3.43	3.43	4.25	4.25	0.75	0.75
7.78	7.78	1.65	1.65	11.63	17.37
10.57	14.28	6.42	15.08	1.38	1.38
7.05	7.05	4.22	9.48	10.53	10.53
2.58	2.58	15.53	15.53	12.13	12.13
7.23	9.68	2.90	2.90	14.58	14.58
6.85	34.58	7.02	7.02	11.82	11.82
32.45	42.35	6.42	6.42	5.52	11.27
8.53	14.57	8.98	8.98	19.65	10.70
31.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	8.87	8.87	10.85	38.07

Table 2: American Football League (N F L) data

If we define the following random variables:

U_1= time to first field goal
U_2 = time to first safety or unconverted touchdown
U_3 = time to first converted touchdown,
then $X_1 = max(U_1, U_3)$, and $X_2 = max(U_2, U_3)$ Therefore, (X_1, X_2) has a similar structure as the Marshall-Olkin bivariate exponential model or the proposed BGG model.

We use this data to obtain the estimate the unknown parameters using our bivariate model. We have taken the initial guesses α_1, α_2, α_3 and λ are all equal to 1. The estimate of α_1, α_2, α_3 and λ become 0.043, 0.528, 1.037 and 7.877 respectively. The corresponding log-likelihood value is 37.414 the 95% confidence intervals of α_1, α_2, α_3 and λ are (0,0.127), (0.244, 0.812), (0.627, 1.447), (5.287, 10.467) respectively.

5.3 Conclusions

In this paper we have proposed bivariate generalized Gompertz distribution function whose marginals are generalized Gompertz distributions. This new bivariate distribution has several interesting properties and it can be used as an alternative to the several absolute continuous bivariate distributions, like Block and Basu bivariate distribution [1]. The generation of random samples from proposed bivariate distribution is very simple, and therefore Monte Carlo simulation can be performed very easily for different statistical inference purpose. It is observed that the MLEs of the unknown parameters can be obtained by solving four non-linear equations and Monte Carlo simulation indicate that the performance of the MLEs are quite satisfactory. Analysis of one real data indicates that the performance of the confidence intervals based on asymptotic distribution.

6. REFERENCES

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