Some Fixed Point Theorems for Weakly Tangential Maps on 
2 - Metric Type Spaces

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ABSTRACT—Here some of the fixed point theorems have been derived for weakly tangential maps from metric type spaces to 2–metric type spaces.

Keywords—Fixed point theorem, 2–Metric type spaces, Self- maps, Compatible and weakly tangential maps.

1. INTRODUCTION

Khamsi introduced a metric type space which is a generalization of a metric spaces[3]. Also, he proved some properties of metric type spaces and some fixed point theorems for a self-map on a metric type space.

The concept of occasionally weakly compatible mappings in metric space was introduced by Al. Thagafi and Shahzad [2]. Moreover Akkouchi introduced weakly tangential maps studied the well-posedness of the common fixed point problem for two weakly tangential self- maps on a metric space[1] and we extend the same to 2–metric type spaces.

2. PRELIMINARIES

Definition 2.1[9] :
A 2–metric space is a set X with a real valued nonnegative function
σ : X × X × X → [0, ∞) such that
i) for any x, y ϵ X, ( x ≠ y), there exists a point z ϵ X such that σ(x, y, z) ≠ 0
ii) σ(x, y, z) = 0 if at least two of the points x, y, z coincide.
iii) σ(x, y, z) = σ(x, z, y) = σ(y, z, x) = σ(y, x, z) (Symmetry)
iv) σ(x, y, z) ≤ σ(x, y, w) + σ(x, w, z) + σ(w, y, z), for all x, y, z, w ϵ X (Tetrahedron inequality)
The function σ is called 2-metric and (X, σ) is called a 2-metric space.

Definition 2.2[10] :
Let X be a nonempty set, K ≥ 1 be a real number and σ : X × X × X → [0, ∞) satisfy the following properties
i) for any x, y ϵ X, ( x ≠ y), there exists a point z ϵ X such that σ(x, y, z) ≠ 0
ii) σ(x, y, z) = 0 if at least two of the points x, y, z coincide.
iii) σ(x, y, z) = σ(x, z, y) = σ(y, z, x) = σ(y, x, z) (Symmetry)
iv) σ(x, y, z) ≤ K[σ(x, y, u) + σ(x, u, z) + σ(u, y, z)], for all x, y, z , u ϵ X.
then (X, σ, K) is called 2-metric type space.
For K = 1, 2–metric type space is simply a 2–metric space.
A 2–metric type space may satisfy the following additional property:
v) Function σ is continuous in two variables, that is xₙ → x, yₙ → y in (X, σ, K) imply
σ(xₙ, yₙ, z) → σ(x, y, z)
Definition 2.3 [3]:
Let \((X, \sigma, K)\) be 2 – metric type space.

(i) The sequence \(\{x_n\}\) converges to \(x \in X\) if and only if 
\[
\lim_{n \to \infty} \sigma(x_n, x, z) = 0.
\]

(ii) The sequence \(\{x_n\}\) is Cauchy sequence if and only if 
\[
\lim_{n, m \to \infty} \sigma(x_n, x_m, z) = 0.
\]

(iii) \((X, \sigma, K)\) is complete if and only if every Cauchy sequence in \(X\) is convergent.

Lemma 2.4 :
Let \((X, \sigma, K)\) be a 2 – metric type space and \(\{x_n\}, \{y_n\}\) be two sequences in \(X\). If \(\{x_n\}\) is a Cauchy sequence an 
\[
\lim_{n \to \infty} \sigma(x_n, y_n, z) = 0,
\]
then \(\{y_n\}\) is a Cauchy sequence. Furthermore, if \(x_n \to u\) then \(y_n \to u\).

\textbf{Proof:} For all \(n, m \in \mathbb{N}\), it follows from tetrahedron inequality 
\[
\sigma(y_n, y_m, z) \leq K[\sigma(y_n, x_n, z) + \sigma(x_n, y_m, z) + \sigma(y_n, y_m, x_n)],
\]
Applying the limit as \(n, m \to \infty\) then \(\sigma(y_n, y_m, z) = 0\), that is \(\{y_n\}\) is a Cauchy sequence.

By using tetrahedron inequality, we have 
\[
\sigma(y_n, u, z) \leq K[\sigma(x_n, u, z) + \sigma(y_n, x_n, z) + \sigma(y_n, u, x_n)]
\]
if \(x_n \to u\) and \(n \to \infty\) in the above inequality, we have 
\[
\lim_{n \to \infty} \sigma(y_n, u, z) = 0. \text{ this implies } y_n \to u.
\]

Definition 2.5.[5] :
Let \(X\) be a nonempty set and \(S, T : X \to X\) be two maps on \(X\).

(i) A point \(u \in X\) is called a coincidence point of \(S\) and \(T\) if \(Su = Tu\).

(ii) \(S\) and \(T\) are said to be occasionally weakly compatible if there exists a point \(u \in X\) which is a coincidence point of \(S\) and \(T\) at which \(S\) and \(T\) commute.

Lemma 2.6:
Let \(X\) be a nonempty set and \(S, T\) be two occasionally weakly compatible self-maps of \(X\). If \(S\) and \(T\) have a unique point \(u = Sx = Tx\), then \(u\) is the unique common fixed point of \(S\) and \(T\).

Definition 2.7 :
Let \((X, \sigma, K)\) be an 2 – metric type space and \(S, T : X \to X\) be two self-maps on \(X\).

i) \(S\) and \(T\) are said to be weakly tangential if there exists a sequence \(\{x_n\}\) in \(X\) such that 
\[
\lim_{n \to \infty} \sigma(Fx_n, Tx_n, z) = 0.
\]

ii) \(\{S, T\}\) is well – posed in the contest of common fixed point theorem is said to be well-posed if

(a) \(S\) and \(T\) have a unique common fixed point \(u \in X\), that is there exists a unique point \(u \in X\) such that 
\(Su = Tu = u\).

(b) For every sequence \(\{x_n\}\) in \(X\), if 
\[
\lim_{n \to \infty} \sigma(x_n, Fx_n, z) = 0 = \lim_{n \to \infty} \sigma(x_n, Tx_n, z) \text{ then } \lim_{n \to \infty} \sigma(x_n, x, z) = 0.
\]

3. MAIN RESULTS

Theorem 3.1:
Let \((X, \sigma, K)\) be an 2-metric type space and \(S, T : X \to X\) be two maps and \(\psi : ([0, \infty) \times [0, \infty)) \to [0, \infty)\) be a function such that 

(i) \(\sigma\) is continuous in each variable;

(ii) \(S(X)\) is a complete subspace of \(X\);

(iii) \(\psi\) is continuous and \(\psi(t, 0) = 0 = \psi(0, t)\) for all \(t \in [0, \infty)\);
(iv) $S$ and $T$ satisfy the following

$$
\sigma(Tx, Ty, z) \leq \frac{1}{K} \left[ b_0 \psi(\sigma(Sx, Tx, z), \sigma(Sx, Ty, z)) + b_1 \sigma(Sx, Sy, z) + b_2(\sigma(Sx, Tx, z) + \sigma(Sy, Ty, z)) + b_3(\sigma(Sx, Ty, z) + \sigma(Sy, Tx, z)) \right]$

for all $x, y, z \in X$ where $b_i = b_i(x, y, z), j = 0, 1, 2, 3$, are non-negative functions from which there exist two constants $M > 0$ and $\lambda \in (0, 1)$ satisfying

$$
b_i(x, y, z) \leq M \quad \text{and} \quad b_i(x, y, z) + b_j(x, y, z) \leq \lambda, \quad \text{for all } x, y, z \in X;
$$

Hence we get $\sigma(y, Tv, z) = 0$. This proves that $v$ is a coincidence point of $S$ and $T$. That is to prove $Sv = Tv$, that is, $\sigma(Sv, Tv, z) = 0$. Which is a contradiction.

Proof: (a)

Since $S$ and $T$ are weakly tangential, there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n \to \infty} \sigma(Fx_n, Tx_n, z) = 0. \quad (3)$$

For all $n \in \mathbb{N}$, put $y_n = Tx_n$ and $u_n = Fx_n$. We shall prove that $\{y_n\}$ and $\{u_n\}$ are Cauchy sequences. we have

$$\sigma(y_n, y_m, z) = \sigma(Tx_n, Tx_m, z) \leq \frac{1}{K} \left[ b_0 \psi(\sigma(Sx_n, Tx_n, z), \sigma(Sx_n, Tx_m, z)) + b_1 \sigma(Sx_n, Sx_m, z) + b_2(\sigma(Sx_n, Tx_n, z) + \sigma(Sx_m, Tx_m, z)) + b_3(\sigma(Sx_n, Ty_n, z) + \sigma(Sx_m, Ty_m, z)) \right] \quad \text{by (1)}$$

Where $b_i = b_i(x, y, z)$ for $j = 0, 1, 2, 3$.

$$\leq \frac{1}{K} \left[ b_0 \psi(\sigma(Sx_n, Tx_n, z), \sigma(Sx_n, Tx_m, z)) + b_1 [\sigma(Sx_n, Tx_n, z) + \sigma(Tx_n, Tm_n, z) + \sigma(Tm_n, Tx_n, z)] \right] + \frac{1}{K} \left[ b_2(\sigma(Sx_n, Tx_n, z) + \sigma(Sx_m, Tx_m, z)) + b_3(\sigma(Sx_n, Ty_n, z) + \sigma(Sx_m, Ty_m, z)) \right]$$

Applying the limit as $n, m \to \infty$ and the fact that $\psi$ is continuous at $(0, 0)$, using (3), we obtain

$$\lim_{n \to \infty} \sigma(y_n, y_m, z) \leq \lim_{n \to \infty} \sigma(y_n, y_m, z) \leq \lambda \quad \text{limit} \quad \lim_{n \to \infty} \sigma(y_n, y_m, z)$$

Which is a contradiction.

Since $\lambda \in [0, 1)$, we get $\{y_n\}$ is a Cauchy sequence.

By Lemma 2.4, we also get $\{u_n\}$ is a Cauchy sequence.

Since $S(X)$ is complete, there exists $y = Sv \in S(X)$ for some $v \in X$ such that

$$\lim_{n \to \infty} y_n = y = Sv. \quad (4)$$

Now we will show that $y$ is the common fixed point of $S$ and $T$. That is to prove $Sv = Tv$, that is, $\sigma(Sv, Tv, z) = 0$. Suppose that $\sigma(Sv, Tv, z) > 0$.

By (1), we have

$$\sigma(Tx_n, Tv, z) \leq \frac{1}{K} \left[ b_0 \psi(\sigma(Sx_n, Tx_n, z), \sigma(Sv, Tv, z)) + b_1 \sigma(Sx_n, Sv, z) + b_2(\sigma(Sx_n, Tx_n, z) + \sigma(Sv, Tv, z)) + b_3(\sigma(Sx_n, Ty_n, z) + \sigma(Sv, Tv, z)) \right]$$

Where $b_i = b_i(x, y, z)$ for $j = 0, 1, 2, 3$.

$$\leq \frac{1}{K} \left[ b_0 \psi(\sigma(Sx_n, Tx_n, z), \sigma(Sv, Tv, z)) + \frac{1}{K} b_1 \sigma(Sv, Sv, z) + \frac{1}{K} b_2(\sigma(Sx_n, Tx_n, z) + \sigma(Sv, Tv, z)) + \frac{1}{K} b_3(\sigma(Sv, Tv, z) + \sigma(Sx_n, Sv, z)) \right] + \frac{1}{K} b_i(\sigma(Sv, Tv, z) + \sigma(Sx_n, Tx_n, z) + \sigma(Sx_n, Sv, Tx_n, z))$$

Applying the limit as $n \to \infty$, we get

$$\sigma(y, Tv, z) \leq \frac{b_3}{K} + 2b_3 \sigma(y, Tv, z) \leq \lambda \sigma(y, Tv, z)$$

This is a contradiction.

Hence we get $\sigma(y, Tv, z) = 0$.

i.e. $\sigma(Sv, Tv, z) = 0$. Therefore $y = Sv =Tv$.

This proves that $v$ is a coincidence point of $S$ and $T$. 

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Now we shall prove that if there exists \( w \in X \) with \( w = Su = Tu \) for some \( u \in X \), then \( w = y \).

By applying (1), we have
\[
\sigma(y, w, z) = \sigma(Tv, Tu, z) \\
\leq \frac{1}{k} \left[ b_0 \psi(\sigma(Sv, Tv, z), \sigma(Su, Tu, z)) + b_1 \sigma(Sv, Su, z) + b_2(\sigma(Sv, Tv, z) + \sigma(Su, Tu, z)) \\
+ b_3(\sigma(Sv, Tu, z) + \sigma(Su, Tv, z)) \right] \\
\leq \frac{1}{k}(b_1 + 2b_3) \sigma(y, w, z) \leq \lambda \sigma(y, w, z),
\]

Since \( \lambda \in [0, 1) \), we get \( \sigma(y, w, z) = 0 \), that is \( y = w \). By Lemma 2.6, we have \( y \) is the unique common fixed point of \( S \) and \( T \).

**Proof: (b)**

Let \( y \) be the unique common fixed point of \( F \) and \( T \). For each sequence \( \{x_n\} \) in \( X \) with
\[
\lim_{n \to \infty} \sigma(x_n, Sx_n, z) = 0 = \lim_{n \to \infty} \sigma(x_n, Tx_n, z)
\]
(5)

To prove
\[
\lim_{n \to \infty} \sigma(x_n, y, z) = 0.
\]

\[0 \leq \sigma(Tx_n, Sx_n, z) \leq 1/K[\sigma(Tx_n, Sx_n, x_n) + \sigma(Tx_n, x_n, z) + \sigma(x_n, Sx_n, z)]\]

Applying the limit as \( n \to \infty \) and using (5), we get
\[
\lim_{n \to \infty} \sigma(Tx_n, Sx_n, z) = 0.
\]
(6)

As in the proof of (a),

For each \( n \in \mathbb{N} \), put \( y_n = Tx_n \) and \( w_n = Fx_n \), then \( \{y_n\} \) and \( \{w_n\} \) are two Cauchy sequences, since \( S(X) \) is Complete, there exists \( x = Fv \) for some \( v \in X \) such that \( \lim_{n \to \infty} w_n = x \).

By (6) and Lemma 2.4, we have
\[
\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = x.
\]
(7)

As in the proof of (a), \( x \) must be the unique common fixed point of \( S \) and \( T \). It implies that \( x = y \).

From (5), (6) and Lemma 2.4, we have \( x_n \to y \), that is \( \lim_{n \to \infty} \sigma(x_n, y, z) = 0. \)

**Proof: (c)**

Let \( y \) be the unique common fixed point of \( S \) and \( T \). For each sequence \( \{u_n\} \) with
\[
\lim_{n \to \infty} u_n = y = Sy = Ty, \quad \text{we need to prove} \quad \lim_{n \to \infty} Tu_n = Ty. \quad \text{By using (1)}
\]
\[
\sigma(Tu_n, y, z) = \sigma(Tu_n, Ty, z) \\
\leq \frac{1}{k} \left[ b_0 \psi(\sigma(Su_n, Tu_n, z), \sigma(Sy, Ty, z)) + b_1 \sigma(Su_n, Sy, z) + b_2(\sigma(Su_n, Tu_n, z) + \sigma(Sy, Ty, z)) \\
+ b_3(\sigma(Su_n, Ty, z) + \sigma(Sy, Tu_n, z)) \right] \quad \text{where } b_j = b_j(u_n, y, z) \text{ for } j = 0, 1, 2, 3.
\]
\[
\leq \lambda \sigma(Fu_n, y, z) + \lambda \sigma(y, Tu_n, z),
\]
This imply \( 0 \leq (1 - \lambda) \sigma(Tu_n, y, z) \leq \lambda \sigma(Su_n, y, z) \).

Applying the limit as \( n \to \infty \) and using the continuity of \( S \) at \( y \),
we obtain \( \lim_{n \to \infty} \sigma(Tu_n, y, z) = 0 \). This implies \( \lim_{n \to \infty} Tu_n = y \) i.e \( \lim_{n \to \infty} Tu_n = Ty \).

**Corollary 3.2 [4]:**

Let \((X, \sigma, K)\) be an \(2\) – metric type space and \(S, T : X \to X\) be two self- maps such that

1. \(\sigma\) is continuous in all variable;
2. \(S(X)\) is a complete subspace of \(X\);
3. \(S\) and \(T\) satisfy the following
   \[
   \sigma(Tx, Ty, z) \leq \frac{p}{R} [\sigma(Sx, Ty, z) + \sigma(Sy, Ty, z)] + \frac{q}{R} [\sigma(Sx, Tx, z) + \sigma(Sx, Tx, z)]
   \]
   (8)
   for all \(x, y, z \in X\) and for some \(p, q, r \geq 0, p + q + 2r \in [0, 1]\);
4. \(S\) and \(T\) are weakly tangential and occasionally weakly compatible.

Then we have

(a) \(S\) and \(T\) have a unique common fixed point in \(X\);
(b) The common fixed point problem of \(\{S, T\}\) is said to be well-posed;
(c) If \(S\) is continuous at the unique common fixed point, then \(T\) is continuous at the unique common fixed point.

**Proof:**

Put \(\psi(s, t) = 0\) for all \(s, t \in [0, 1)\) and \(b_0 = 0, b_1 = p, b_2 = q, b_3 = r, M = 1, \lambda = p + q + 2r\), we get the conclusion by using Theorem 3.1.

**Corollary 3.3 [7]:**

Let \(T\) be a self map defined on complete \(2\)-metric type space \((X, \sigma, K), \sigma\) is continuous in each variable and \(\sigma(Tx, Ty, z) \leq \lambda \sigma(x, y, z)\) for some \(\lambda \in [0, 1)\) and all \(x, y, z \in X\). Then \(T\) has a unique fixed point in \(X\). Moreover, \(T\) is continuous at the fixed point.

**Proof:** By choosing \(S\) is the identity map and \(q = r = 0\) in Corollary 3.2, we get the conclusion.

4. REFERENCES