Some Fixed Point Theorems for Weakly Tangential Maps on 2 - Metric Type Spaces

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ABSTRACT— ***Here some of the fixed point theorems have been derived for weakly tangential maps from metric type spaces to 2 – metric type spaces.***

Keywords— Fixed point theorem, 2 – Metric type spaces, Self- maps, Compatible and weakly tangential maps.

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1. INTRODUCTION

Khamsi introduced a metric type space which is a generalization of a metric spaces[3]. Also, he proved some properties of metric type spaces and some fixed point theorems for a self-map on a metric type space.

The concept of occasionally weakly compatible mappings in metric space was introduced by Al. Thagafi and Shahzad [2]. Moreover Akkouchi introduced weakly tangential maps studied the well-posedness of the common fixed point problem for two weakly tangential self- maps on a metric space[1] and we extend the same to 2 – metric type spaces.

2. PRELIMINARIES

Definition 2.1[9] :

A 2 – metric space is a set X with a real valued nonnegative function
$\sigma : X \times X \times X \to [0, \infty)$ such that

i) for any x, y ϵ X, ($x \neq y$), there exists a point z ϵ X such that $\sigma(x, y, z) \neq 0$
ii) $\sigma(x, y, z) = 0$ if at least two of the points x, y, z coincide.
iii) $\sigma(x, y, z) = \sigma(x, z, y) = \sigma(y, z, x) = \sigma(y, x, z)$ (Symmetry)
iv) $\sigma(x, y, z) \leq \sigma(x, y, w) + \sigma(x, w, z) + \sigma(w, y, z)$, for all x, y, z, w ϵ X (Tetrahedron inequality)

The function σ is called 2-metric and (X, σ) is called a 2-metric space.

Definition 2.2[10] :

Let X be a nonempty set, $K \geq 1$ be a real number and $\sigma : X \times X \times X \to [0, \infty)$ satisfy the following properties

i) for any x, y ϵ X, ($x \neq y$), there exists a point z ϵ X such that $\sigma(x, y, z) \neq 0$
ii) $\sigma(x, y, z) = 0$ if at least two of the points x, y, z coincide.
iii) $\sigma(x, y, z) = \sigma(x, z, y) = \sigma(y, z, x) = \sigma(y, x, z)$ (Symmetry)
iv) $\sigma(x, y, z) \leq K[\sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z)]$, for all x, y, z , u ϵ X.

then (X, σ, K) is called 2-metric type space.

For K = 1, 2 – metric type space is simply a 2 – metric space.

A 2 – metric type space may satisfy the following additional property:

v) Function σ is continuous in two variables, that is $x_n \to x$, $y_n \to y$ in (X, σ, K) imply $\sigma(x_n, y_m, z) \to \sigma(x, y, z)$

Definition 2.3 [3] :

Let (X, σ, K) be 2 – metric type space.

(i) The sequence $\{x_n\}$ converges to x ∈ X if and only if $\lim_{n\to\infty} \sigma(x_n, x, z) = 0$.

(ii) The sequence $\{x_n\}$ is Cauchy sequence if and only if $\lim_{n,m\to\infty} \sigma(x_n, x_m, z) = 0$.

(iii) (X, σ, K) is complete if and only if every Cauchy sequence in X is convergent.

Lemma 2.4 :

Let (X, σ, K) be a 2 – metric type space and $\{x_n\}$, $\{y_n\}$ be two sequences in X. If $\{x_n\}$ is a Cauchy sequence an $\lim_{n\to\infty} \sigma(x_n, y_n, z) = 0$, then $\{y_n\}$ is a Cauchy sequence. Furthermore, if $x_n \to u$ then $y_n \to u$.

Proof: For all n, m ∈ℕ, it follows from tetrahedron inequality

$\sigma(y_n, y_m, z) \leq K[\sigma(y_n, x_n, z) + \sigma(x_n, y_m, z) + \sigma(y_n, y_m, x_n)]$,

Applying the limit as n, m → ∞ then $\sigma(y_n, y_m, z) = 0$, that is $\{y_n\}$ is a Cauchy sequence.

By using tetrahedron inequality, we have

$\sigma(y_n, u, z) \leq K[\sigma(x_n, u, z) + \sigma(y_n, x_n, z) + \sigma(y_n, u, x_n)]$

if $x_n \to u$ and $n \to \infty$ in the above inequality, we have

$\lim_{n\to\infty} \sigma(y_n, u, z) = 0$. this implies $y_n \to u$.

Definition 2.5.[5] :

Let X be a nonempty set and S, T : X → X be two maps on X.

(i) A point u ∈ X is called a coincidence point of S and T if Su = Tu.

(ii) S and T are said to be occasionally weakly compatible if there exists a point u ∈ X which is a coincidence point of S and T at which S and T commute.

Lemma 2.6:

Let X be a nonempty set and S, T be two occasionally weakly compatible self-maps of X. If S and T have a unique point u = Sx = Tx, then u is the unique common fixed point of S and T.

Definition 2.7 :

Let (X, σ, K) be an 2 – metric type space and S, T : X → X be two self-maps on X.

i) S and T are said to be weakly tangential if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \sigma(Fx_n, Tx_n, z) = 0$.

ii) {S, T} is well – posed in the contest of common fixed point theorem is said to be well-posed if

(a) S and T have a unique common fixed point u ∈ X, that is there exists a unique point u ∈ X such that Su = Tu = u.

(b) For every sequence $\{x_n\}$ in X, if

$$\lim_{n\to\infty} \sigma(x_n, Fx_n, z) = 0 = \lim_{n\to\infty} \sigma(x_n, Tx_n, z) \text{ then } \lim_{n\to\infty} \sigma(x_n, x, z) = 0.$$

3. MAIN RESULTS

Theorem 3.1:

Let (X, σ, K) be an 2-metric type space and S, T : X → X be two maps and ψ : ([0, ∞) × [0, ∞)) → [0, ∞) be a function such that

(i) σ is continuous in each variable;

(ii) S(X) is a complete subspace of X;

(iii) ψ is continuous and ψ (t, 0) = 0 = ψ (0, t) for all t ∈ [0, ∞);

(iv) S and T satisfy the following

$$\sigma(Tx, Ty, z) \le \frac{1}{K}[b_0\psi(\sigma(Sx, Tx, z), \sigma(Sx, Ty, z)) + b_1\sigma(Sx, Sy, z) + b_2(\sigma(Sx, Tx, z) + \sigma(Sy, Ty, z)) + b_3(\sigma(Sx, Ty, z) + \sigma(Sx, Tx, z)) \quad (1)$$

for all x, y, z ∈ X where $b_j = b_j(x, y, z)$, j = 0, 1, 2, 3, are non-negative functions from which there exist two constants M > 0 and λ ∈ [0, 1) satisfying

$b_0(x, y, z) \le M$ and

$b_1(x, y, z) + b_2(x, y, z) + 2b_3(x, y, z) \le \lambda$, for all x, y, z ∈ X; (2)

(v) S and T are weakly tangential and occasionally weakly compatible. Then we have

- **(a)** S and T have a unique common fixed point in X;
- **(b)** {S, T} is well- posed in the contest of common fixed point theorem
- **(c)** If S is continuous at the unique common fixed point, then T is continuous at the unique common fixed point.

Proof: (a)

Since S and T are weakly tangential, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \sigma(Fx_n, Tx_n, z) = 0. \quad (3)$$

For all n ∈ℕ, put $y_n = Tx_n$ and $u_n = Fx_n$. We shall prove that $\{y_n\}$ and $\{u_n\}$ are Cauchy sequences.

we have

$$\sigma(y_n, y_m, z) = \sigma(Tx_n, Tx_m, z)$$
$$\le \frac{1}{K}[b_0\,\psi(\sigma(Sx_n, Tx_n, z), \sigma(Sx_m, Tx_m, z)) + b_1\,\sigma(Sx_n, Sx_m, z) + b_2(\sigma(Sx_n, Tx_n, z) + \sigma(Sx_m, Tx_m, z)) + b_3(\sigma(Sx_n, Tx_m, z) + \sigma(Sx_m, Tx_n, z))];\ \text{by (1)}$$

Where $b_j = b_j(x_n, x_m, z)$ for j = 0, 1, 2, 3.

$$\le \frac{1}{K} b_0\,\psi(\sigma(Sx_n, Tx_n, z), \sigma(Sx_m, Tx_m, z)) + b_1[\sigma(Sx_n, Tx_n, z) + \sigma(Tx_n, Tx_m, z) + \sigma(Sx_n, Tx_m, Tx_n)]$$
$$+\frac{1}{K}b_2[(\sigma(Sx_n, Tx_n, z) + \sigma(Sx_m, Tx_m, z)] + b_3[\sigma(Sx_n, Tx_n, z) + \sigma(Sx_n, Tx_m, Tx_n) + \sigma(Tx_n, Tx_m, z) + \sigma(Sx_m, Tx_m, z) + \sigma(Sx_m, Tx_n, Tx_m) + \sigma(Tx_m, Tx_n, z)]$$

Applying the limit as n, m → ∞ and the fact that ψ is continuous at (0, 0), using (3),

we obtain $\lim_{n\to\infty} \sigma(y_n, y_m, z) \le (b_1 + 2b_3) \lim_{n\to\infty} \sigma(y_n, y_m, z) \le \lambda \lim_{n\to\infty} \sigma(y_n, y_m, z)$

Which is a contradiction.

Since λ ∈ [0, 1), $\lim_{n\to\infty} \sigma(y_n, y_m, z) = 0.$ this proves that $\{y_n\}$ is a Cauchy sequence.

By Lemma 2:4, we also get $\{u_n\}$ is a Cauchy sequence.

Since S(X) is complete, there exists y = Sv ∈ S(X) for some v ∈ X such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} w_n = y = Sv. \quad (4)$$

Now we will show that y is the common fixed point of S and T. that is to prove Sv = Tv, that is, σ(Sv, Tv, z) = 0.

Suppose that σ(Sv, Tv, z) > 0.

By (1), we have

$$\sigma(Tx_n, Tv, z) \le \frac{1}{K}[b_0\psi(\sigma(Sx_n, Tx_n, z), \sigma(Sv, Tv, z)) + b_1\,\sigma(Sx_n, Sv, z) + b_2(\sigma(Sx_n, Tx_n, z) + \sigma(Sv, Tv, z)) + b_3(\sigma(Sx_n, Tv, z) + \sigma(Sv, Tx_n, z))] \quad \text{where } b_j = b_j(x_n, v, z) \text{ for } j = 0, 1, 2, 3.$$

$$\le \frac{1}{K} b_0\,\psi(\sigma(Sx_n, Tx_n, z), \sigma(Sv, Tv, z)) + \frac{1}{K}b_1\sigma(Sx_n, Sv, z) + \frac{1}{K}b_2[(\sigma(Sx_n, Tx_n, z) + \sigma(Sv, Tv, z)] + b_3[\sigma(Sv, Tv, z) + \sigma(Sx_n, Tv, Tx_n) + \sigma(Tv, Sx_n, z) + \sigma(Sx_n, Tx_n, z) + \sigma(Sx_n, Sv, Tx_n) + \sigma(Sx_n, Sv, z)]$$

Applying the limit as n → ∞, we get

$$\sigma(y, Tv, z) \le (\frac{b_2}{K} + 2b_3)\,\sigma(y, Tv, z) \le \lambda\,\sigma(y, Tv, z)$$

This is a contradiction.

Hence we get σ(y, Tv, z) = 0.

i.e. σ(Sv, Tv, z) = 0. Therefore y = Sv = Tv.

This proves that v is a coincidence point of S and T.

Now we shall prove that if there exists w ∈ X with w = Su = Tu for some u ∈ X, then w = y.

By applying (1), we have

$$\sigma(y, w, z) = \sigma(Tv, Tu, z)$$
$$\leq \frac{1}{K}[b_0\, \psi(\sigma(Sv, Tv, z), \sigma(Su, Tu, z)) + b_1\, \sigma(Sv, Su, z) + b_2(\sigma(Sv, Tv, z) + \sigma(Su, Tu, z))$$
$$+ b_3(\sigma(Sv, Tu, z) + \sigma(Su, Tv, z))]$$
$$\leq \frac{1}{K}(b_1 + 2b_3)\, \sigma(y, w, z) \leq \lambda\sigma(y, w, z).$$

Since λ ∈ [0, 1), we get σ(y, w, z) = 0, that is y = w. By Lemma 2.6, we have y is the unique common fixed point of S and T.

Proof: (b)

Let y be the unique common fixed point of F and T. For each sequence $\{x_n\}$ in X with

$$\lim_{n \to \infty} \sigma(x_n, Sx_n, z) = 0 = \lim_{n \to \infty} \sigma(x_n, Tx_n, z) \tag{5}$$

To prove

$$\lim_{n \to \infty} \sigma(x_n, y, z) = 0.$$

$0 \leq \sigma(Tx_n, Sx_n, z) \leq 1/K[\sigma(Tx_n, Sx_n, x_n) + \sigma(Tx_n, x_n, z) + \sigma(x_n, Sx_n, z)]$

Applying the limit as n → ∞ and using (5), we get

$$\lim_{n \to \infty} \sigma(Tx_n, Sx_n, z) = 0. \tag{6}$$

As in the proof of (a),

For each n ∈ℕ, put $y_n = Tx_n$ and $w_n = Fx_n$, then $\{y_n\}$ and $\{w_n\}$ are two Cauchy sequences, since S(X) is Complete, there exists x = Fv for some v ∈ X such that $\lim_{n \to \infty} w_n = x.$

By (6) and Lemma 2:4, we have

$$\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = x. \tag{7}$$

As in the proof of (a), x must be the unique common fixed point of S and T. It implies that x = y.

From (5), (6) and Lemma 2:4, we have $x_n \to y$, that is $\lim_{n \to \infty} \sigma(x_n, y, z) = 0.$

Proof: (c)

Let y be the unique common fixed point of S and T. For each sequence $\{u_n\}$ with

$\lim_{n \to \infty} u_n = y = Sy = Ty$, we need to prove $\lim_{n \to \infty} Tu_n = Ty$. By using (1)

$$\sigma(Tu_n, y, z) = \sigma(Tu_n, Ty, z)$$
$$\leq \frac{1}{K}[b_0\psi(\sigma(Su_n, Tu_n, z), \sigma(Sy, Ty, z)) + b_1\, \sigma(Su_n, Sy, z) + b_2(\sigma(Su_n, Tu_n, z) + \sigma(Sy, Ty, z))$$
$$+ b_3(\sigma(Su_n, Ty, z) + \sigma(Sy, Tu_n, z))] \qquad \text{Where } b_j = b_j(u_n, y, z) \text{ for } j = 0, 1, 2, 3.$$
$$\leq \lambda\sigma(Fu_n, y, z) + \lambda\sigma(y, Tu_n, z),$$

This imply $0 \leq (1 - \lambda)\, \sigma(Tu_n, y, z) \leq \lambda\sigma(Su_n, y, z)$.

Applying the limit as n →∞ and using the continuity of S at y, we obtain $\lim_{n \to \infty} \sigma(Tu_n, y, z) = 0$. This implies $\lim_{n \to \infty} Tu_n = y$. i.e $\lim_{n \to \infty} Tu_n = Ty$.

Corollary 3.2 [4] :
Let (X, σ, K) be an 2 – metric type space and S, T : X → X be two self- maps such that

(i) σ is continuous in all variable;
(ii) S(X) is a complete subspace of X;
(iii) S and T satisfy the following

$$\sigma(Tx, Ty, z) \leq \frac{p}{K} \sigma(Sx, Sy, z) + \frac{q}{K} [\sigma(Sx, Tx, z) + \sigma(Sy, Ty, z)] + \frac{r}{K} [\sigma(Sx, Ty, z) + \sigma(Sx, Tx, z))] \tag{8}$$

for all x, y, z ∈ X and for some p, q, r ≥ 0, p + q + 2r ∈ [0, 1);

(iv) S and T are weakly tangential and occasionally weakly compatible.

Then we have

(a) S and T have a unique common fixed point in X;
(b) The common fixed point problem of {S, T} is said to be well-posed;
(c) If S is continuous at the unique common fixed point, then T is continuous at the unique common fixed point.

Proof:

Put ψ (s, t) = 0 for all s, t ∈ [0, 1) and $b_0 = 0$, $b_1 = p$, $b_2 = q$, $b_3 = r$, M = 1, λ = p + q + 2r, we get the conclusion by using Theorem 3.1.

Corollary 3.3 [7]:
Let T be a self map defined on complete 2-metric type space (X, σ, K), σ is continuous in each variable and σ(Tx, Ty, z) ≤ λ σ(x, y, z) for some λ ∈ [0, 1) and all x, y, z in X. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof: By choosing S is the identity map and q = r = 0 in Corollary 3.2, we get the conclusion.

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