On a Fuzzy Completely Closed Filter with Respect of Element in a BH-algebra

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ABSTRACT—In this paper, we introduce a new notion that we call a fuzzy completely closed filter with respect of an element in a BH-algebra, and we link this notion with notions filter and ideal of BH-algebra. We give some properties of fuzzy completely closed filter with respect of an element and we study properties of it.

Keywords—fuzzy completely closed filter with respect of an element, BH-algebra, fuzzy filter, fuzzy ideal, fuzzy completely closed filter and fuzzy completely closed filter.

1. INTRODUCTION


2. PRELIMINARIES

In this section, we review some basic definitions and notations of BH-algebras, fuzzy completely closed filter, filter, ideals and other notions, that we need in our work.

Definition (1.1) [12]:
Let X be a non-empty set. A fuzzy set A in X (a fuzzy subset of X) is a function from X into the closed interval [0,1] of the real number.

Definition (1.2) [8]:
Let A and B be two fuzzy sets in X, then :
1. \((A\cap B)(x)=\min\{A(x),B(x)\}\), for all \(x\in X\).
2. \((A\cup B)(x)=\max\{A(x),B(x)\}\), for all \(x\in X\).

A\cap B and A\cup B are fuzzy sets in X.
In general, if \( \{A_\alpha, \alpha \in \Lambda\} \) is a family of fuzzy sets in \( X \), the:

\[
\bigcup_{i \in \Gamma} A_i(x) = \inf \{A_i(x), i \in \Gamma\}, \text{ for all } x \in X \text{ and }
\]

\[
\bigcap_{i \in \Gamma} A_i(x) = \sup \{A_i(x), i \in \Gamma\}, \text{ for all } x \in X.
\]

which are also fuzzy sets in \( X \).

**Definition (1.3) [5]:**

A BH-algebra is a nonempty set \( X \) with a constant \( 0 \) and a binary operation \( \ast \) satisfying the following conditions:

1) \( x \ast x = 0, \forall x \in X \).
2) \( x \ast y = 0 \) and \( y \ast x = 0 \) imply \( x = y, \forall x, y \in X \).
3) \( x \ast 0 = x, \forall x \in X \).

**Example (1.4)[4]:**

Let \( X = \{0, 1, 2\} \) be a set with the following table:

<table>
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<tr>
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<th>0</th>
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Then \( (X, \ast, 0) \) is a BH-algebra.

**Definition (1.5)[9]:**

A BH-algebra \( X \) is called an associative BH-algebra if:

\( (x \ast y) \ast z = x \ast (y \ast z) , \) for all \( x, y, z \in X \).

**Definition (1.6)[9]:**

Let \( X \) be a BH-algebra and \( b \in X \), a filter \( F \) is called a completely closed filter with respect to \( b \) (denoted by \( b \)-completely closed filter) if \( b \ast (x \ast y) \in F, \forall x, y \in F \)

**Definition (1.7)[19]:**

A fuzzy set \( M \) in a BH-algebra \( X \) is said to be fuzzy normal if it satisfies the inequality:

\[
M((x \ast a) \ast (y \ast b)) \geq \min\{M(x \ast y), M(a \ast b)\}, \forall a, b, x, y \in X.
\]

**Definition (1.8) [14]:**

A fuzzy set \( A \) in a BH-algebra \( X \) is said to be a fuzzy subalgebra of \( X \) if it satisfies:

\[
A(x \ast y) \geq \min\{A(x), A(y)\}, \forall x, y \in X.
\]

**Definition (1.9)[6]:**

A non constant fuzzy set \( A \) of \( X \) is a fuzzy filter if

1. \( A(x \ast y) \geq \min\{A(x), A(y)\} \) and \( A(y \ast x) \geq \min\{A(x), A(y)\} \) For any \( x, y \in X \).
2. \( A(y) \geq A(x) \), when \( x \leq y \).

**Definition (1.10)[9]:**

Let \( X \) be a BH-algebra, \( A \) be a fuzzy filter of \( X \) and \( b \in X \). Then \( A \) is called a fuzzy closed filter with respect to an element \( b \in X \), denoted by a fuzzy \( b \)-closed filter of \( X \), if \( A(b \ast (0 \ast x)) \geq A(x), \forall x \in X \).

**Definition (1.11)[9]:**

Let \( X \) be a BH-algebra and \( A \) be a fuzzy filter of \( X \). Then \( A \) is called a fuzzy completely closed filter, if \( A(x \ast y) \geq \min\{A(x), A(y)\} \) \( \forall x, y \in X \).
Definition (1.12) [5]:

A fuzzy subset $A$ of a BH-algebra $X$ is said to be a fuzzy ideal if and only if:

1) For any $x \in X$, $A(0) \geq A(x)$.

2) For any $x, y \in X$, $A(x) \geq \min\{A(x*y), A(y)\}$.

Definition (1.13) [9]:

Let $X$ be a BH-algebra and $A$ be a fuzzy ideal of $X$. Then $A$ is called a fuzzy completely closed ideal if $A(x*y) \geq \min\{A(x), A(y)\}, \forall x, y \in X$.

Definition (1.14) [9]:

Let $X$ be a BH-algebra and $A$ be a fuzzy ideal of $X$. Then $A$ is called a fuzzy completely closed ideal with respect to an element $b \in X$, denoted by a fuzzy $b$-completely closed ideal of $X$, if $A(b*(x*y)) \geq \min\{A(x), A(y)\}, \forall x, y \in X$.

Theorem (1.15) [9]:

Let $X$ be a BH-algebra and $A$ be a fuzzy set. Then $A$ is a fuzzy filter if and only if $A(x) = A(x) + 1 - A(0)$ is a fuzzy filter.

Theorem (1.16) [9]:

Let $X$ be an associative BH-algebra. Then every fuzzy normal set of $X$ is a fuzzy filter.

Proposition (1.17) [9]:

Let $X$ be a BH-algebra and $A$ be a fuzzy completely closed filter then $A(0) \geq A(x), \forall x \in X$.

Proposition (1.18) [9]:

Let $X$ be a BH-algebra. If $M$ is a fuzzy normal set, then $M(0) \geq M(x), \forall x \in X$.

3. MAIN RESULTS

In this section, we define the a fuzzy completely closed filter with respect to an element, and link the notion with another notions in BH-algebra.

Definition (2.1): Let $X$ be a BH-algebra and $A$ be a fuzzy filter of $X$. Then $A$ is called a fuzzy completely closed filter with respect to an element $b \in X$, denoted by a fuzzy $b$-completely closed filter of $X$, if $A(b*(x*y)) \geq \min\{A(x), A(y)\}, \forall x, y \in F$.

Example (2.2): Let $X = \{1, 2, 3\}$ be a BH-algebra, with the following table:

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The fuzzy filter $A$ which is defined by $A(x) =\begin{cases} 0.6 & x = 0.2 \\ 0.3 & x = 1 \end{cases}$ is a fuzzy 2-completely closed filter of $X$, since

$A(2*(0*0)) = A(2*0) = A(2) = 0.6 \geq \min\{A(0), A(0)\} = 0.6$

$A(2*(0*1)) = A(2*1) = A(2) = 0.6 \geq \min\{A(0), A(1)\} = 0.3$

$A(2*(0*2)) = A(2*2) = A(0) = 0.6 \geq \min\{A(0), A(2)\} = 0.6$

$A(2*(1*0)) = A(2*1) = A(2) = 0.6 \geq \min\{A(1), A(0)\} = 0.3$

$A(2*(1*1)) = A(2*0) = A(2) = 0.6 \geq \min\{A(1), A(1)\} = 0.3$

$A(2*(1*2)) = A(2*1) = A(2) = 0.6 \geq \min\{A(1), A(2)\} = 0.3$

$A(2*(2*0)) = A(2*2) = A(0) = 0.6 \geq \min\{A(2), A(0)\} = 0.6$

$A(2*(2*1)) = A(2*2) = A(0) = 0.6 \geq \min\{A(2), A(1)\} = 0.3$

$A(2*(2*2)) = A(2*0) = A(2) = 0.6 \geq \min\{A(2), A(2)\} = 0.6$
Theorem (2.3): Let X be BH-algebra such that if \(x*y=0\) implies \(x=y\) \(\forall x,y \in X\). Then every fuzzy b-completely closed filter is a fuzzy filter.

Proof: Let A be a fuzzy b-completely closed filter and \(x,y \in X\).

1) \(A(x*(x*y)) \geq \min\{A(x),A(y)\}\) [Since A is a fuzzy b-completely closed filter]

Similarly, \(A(y*(y*x)) \geq \min\{A(x),A(y)\}\)

2) Let \(x \leq y\)

\(\Rightarrow x*y=0 \Rightarrow x=y\)

\(\Rightarrow A(y)=A(x) \geq A(x)\)

\(\therefore A\) is a fuzzy filter.

Theorem (2.4): Let X be an associative BH-algebra. Then every fuzzy b-completely closed filter is a fuzzy b-closed filter.

Proof: Let A be a fuzzy b-completely closed filter

Now, \(0,x \in F; b \in X\)

\(A(b*(0*x)) \geq \min\{A(0),A(x)\}\) [A is fuzzy b-completely closed filter, definition (2.1)]

\(A(b*(x*y)) \geq A(x)\) [\(A(0) \geq A(x) \forall x \in X, Proposition\ (1.17)\)]

\(\therefore A\) is a fuzzy b-closed filter.

Proposition (2.5): Let X be an associative BH-algebra. Then every fuzzy normal set \(\forall b \in X, s.t (M(b)=M(0))\) is a fuzzy b-completely closed filter.

Proof: Let M be a fuzzy normal set.

\(\Rightarrow M\) is a fuzzy filter [ (1.16) ]

Now, Let \(x,y \in M, b \in X\)

\(M(b*(x*y))=M(b*((x*y)*(0*0)))\)

\(=M(b*(x*y)*(0*0))\)

\(\geq \min\{M(b),M((x*y))\}\) [Since M is a fuzzy normal set, definition (1.7)]

\(\geq \min\{M(0),M((x*y))\}\) [\(M(b)=M(0)\)]

\(=M((x*y))\)

\(\geq \min\{M(x),M(y)\}\) [Since M is a fuzzy normal set definition (1.7)]

\(\therefore M\) is a fuzzy b-completely closed filter.

Proposition (2.6): Let X be a BH-algebra and A is a fuzzy b-completely closed filter. Then \(A\alpha\) is a b-completely closed filter \(\forall \alpha \in (0,1]\).

Proof: Let A be a fuzzy b-completely closed filter.

To prove \(A\alpha\) is a filter,

1) Let \(x, y \in A\alpha\)

\(\Rightarrow A(x) \geq \alpha, A(y) \geq \alpha\)

\(\Rightarrow \min\{A(x),A(y)\} \geq \alpha\).

but \(A(x*(x*y)) \geq \min\{A(x),A(y)\}\) [Since A is a fuzzy filter, definition (1.9)]

\(\Rightarrow A(x*(x*y)) \geq \alpha\).

\(\therefore x*(x*y) \in A\alpha\)

Similarly,

\(y*(y*x) \in A\alpha\)

2) Let \(x \in A\alpha\) and \(x*y=0\)

\(\Rightarrow A(x) \geq \alpha\).
But \( A(y) \geq A(x) \) \quad [\text{Since } A \text{ is a fuzzy filter and } x \leq y, \text{ definition (1.9)}] \\
\Rightarrow A(y) \geq \alpha, \\
\Rightarrow y \in A\alpha \\
\therefore A\alpha \text{ is a filter} \\

Now, Let \( x, y \in F, b \in X, x, y \in A\alpha \) \\
\Rightarrow A(x) \geq \alpha, \\
\Rightarrow A(y) \geq \alpha. \\
\Rightarrow \min\{A(x), A(y)\} \geq \alpha \\
\text{but } A(b^*(x*y)) \geq \min\{A(x), A(y)\} \quad [\text{Since } A \text{ is a fuzzy b-completely closed filter, definition (2.1)}] \\
\Rightarrow A(b^*(x*y)) \geq \alpha \\
\Rightarrow b^*(x*y) \in A\alpha \\
\therefore A\alpha \text{ is a b-completely closed filter } \forall \alpha \in (0, 1]. \blacksquare

**Proposition (2.7):** Let \( X \) be a BH-algebra and \( A \) be a fuzzy b-completely closed filter. Then the set \( X_A = \{ x \in X : A(x) = A(0) \} \) is a b-completely closed filter.

**Proof:** Let \( A \) be a fuzzy filter,

Since \( A(0) = A(0) \) \\
\therefore 0 \in X_A \\
\therefore X_A \text{ is a non-empty set} 

1) Let \( x, y \in X_A \) \\
\Rightarrow A(x) = A(y) = A(0) \\
\Rightarrow \min\{A(x), A(y)\} = A(0) \\
\text{But } A(x*(x*y)) \geq \min\{A(x), A(y)\} = A(0) \quad [\text{Since } A \text{ is a fuzzy filter. definition (1.9)}] \\
\therefore A(x*(x*y)) \geq A(0) \\
\text{But } A(0) \geq A(x*(x*y)) \quad [\text{Since } A \text{ is a fuzzy completely closed filter. definition (1.11)}] \\
\therefore A(x*(x*y)) = A(0) \\
\therefore x*(x*y) \in X_A \\
\text{Similarly,} \\
y*(y*x) \in X_A 

2) Let \( x \in X_A, x \leq y \) \\
\Rightarrow A(y) \geq A(x) = A(0) \quad [\text{Since } A \text{ is a fuzzy filter, definition (1.9)}] \\
\text{But } A(0) \geq A(y), \quad [\text{Since } A \text{ is a fuzzy completely closed filter, definition (1.11)}] \\
\therefore A(y) = A(0) \\
\therefore y \in X_A \\
\therefore X_A \text{ is a filter.} 

Now, Let \( x, y \in F, b \in X_A \) \\
\Rightarrow A(x) = A(y) = A(0) \\
\Rightarrow \min\{A(x), A(y)\} = A(0) \\
\text{But } A(b^*(x*y)) \geq \min\{A(x), A(y)\} = A(0) \quad [\text{Since } A \text{ is a fuzzy b-completely closed filter definition (2.1)}] \\
\therefore A(x*y) \geq A(0)
But \( A(0) \geq A(x*y) \) \hspace{1cm} \text{[Since } A \text{ is a fuzzy completely closed filter, definition (1.11)]} \\
\therefore A(b^{*}(x*y)) = A(0) \\
\therefore b^{*}(x*y) \in X_{A} \\
\therefore X_{A} \text{ is a } b \text{-completely closed filter.} \blacksquare

**Proposition (2.8)**: Let \( \{A_{i}: i \in I\} \) be a family of fuzzy \( b \)-completely closed filter of a BH-algebra \( X \). Then \( \bigcap_{i \in I} A_{i} \) is a fuzzy \( b \)-completely closed filter of \( X \).

**Proof**: 
To prove that \( \bigcap_{i \in I} A_{i} \) is a fuzzy filter,

1. Let \( x, y \in X \).

\[
\left( \bigcap_{i \in I} A_{i} \right)(x*(x*y)) = \inf \{ A_{i}(x*(x*y)), i \in I \} \\
\geq \inf \{ \min \{ A_{i}(x), A_{i}(y) \}, i \in I \} \hspace{1cm} \text{[Since } A_{i} \text{ is a fuzzy filter, } \forall i \in I, \text{ definition (1.9)]} \\
\geq \min \{ \left( \bigcap_{i \in I} A_{i} \right)(x), \left( \bigcap_{i \in I} A_{i} \right)(y) \}
\]

Similarly \( \left( \bigcap_{i \in I} A_{i} \right)(y*(y*x)) \)

2. Let \( x \in X \) and \( x \leq y \)

\[
\Rightarrow \left( \bigcap_{i \in I} A_{i} \right)(x) = \inf \{ A_{i}(x), i \in I \} \\
\geq \inf \{ A_{i}(y), i \in I \} \hspace{1cm} \text{[Since } A_{i} \text{ is a fuzzy filter, } \forall i \in I, \text{ definition (1.9)]}
\]

\[
= \left( \bigcap_{i \in I} A_{i} \right)(y)
\]

\[
\Rightarrow \left( \bigcap_{i \in I} A_{i} \right) \text{ is a fuzzy filter}
\]

To prove that \( \left( \bigcap_{i \in I} A_{i} \right) \) is a fuzzy \( b \)-completely closed filter of \( X \)

Let \( x, y \in F, b \in X \)

\[
\left( \bigcap_{i \in I} A_{i} \right)(b^{*}(x*y)) = \inf \{ A_{i}(b^{*}(x*y)), i \in I \} \\
\geq \inf \{ \min \{ A_{i}(x), A_{i}(y) \}, i \in I \} \\
\geq \min \{ \inf A_{i}(x), \inf A_{i}(y) \}, i \in I \} \\
\geq \min \{ \left( \bigcap_{i \in I} A_{i} \right)(x), \left( \bigcap_{i \in I} A_{i} \right)(y) \}
\]

\[
\Rightarrow \left( \bigcap_{i \in I} A_{i} \right)(b^{*}(x*y)) \geq \min \{ \left( \bigcap_{i \in I} A_{i} \right)(x), \left( \bigcap_{i \in I} A_{i} \right)(y) \} \hspace{0.5cm} \forall x, y \in F
\]

Therefore, \( \left( \bigcap_{i \in I} A_{i} \right) \) is a fuzzy \( b \)-completely closed filter of \( X \). \blacksquare

**Proposition (2.9)**: 
Let \( \{A_{i}: i \in I\} \) be a family of fuzzy \( b \)-completely closed filter of a BH-algebra \( X \). Then \( \bigcup_{i \in I} A_{i} \) is a fuzzy \( b \)-completely closed filter of \( X \).

**Proof**: 
To prove that \( \bigcup_{i \in I} A_{i} \) is a fuzzy filter,

1. Let \( x, y \in X \).

\[
\left( \bigcup_{i \in I} A_{i} \right)(x^{*}(x*y)) = \sup \{ A_{i}(x^{*}(x*y)), i \in I \} \\
\geq \sup \{ \min \{ A_{i}(x), A_{i}(y) \}, i \in I \} \hspace{1cm} \text{[Since } A_{i} \text{ is a fuzzy filter, } \forall i \in I, \text{ definition (1.9)]}
\]

But \( \{A_{i}: i \in I\} \) is a chain \Rightarrow there exist \( j \in \Gamma \) such that
sup{ min{Ai(x*y), Ai(y)}, i ∈ Γ } = min{Aj(x), Aj(y)}

= min{sup{Ai(x), i ∈ Γ}, sup{Ai(y), i ∈ Γ}}

\geq min\{ (\bigcup_{i \in \Gamma} A_i)(x), (\bigcup_{i \in \Gamma} A_i)(y) \}

Similarly \( \bigcup_{i \in \Gamma} A_i(y^*(y*x)) \)

(2) Let \( x \in X \) and \( x \leq y \)

⇒ \( \bigcup_{i \in \Gamma} A_i(x) = \sup\{ Ai(x), i \in \Gamma \} \)

\geq \sup\{ Ai(y), i \in \Gamma \} \quad \text{[Since \( Ai \) is a fuzzy filter, \( \forall i \in \Gamma \), definition (1.9)]}

= \( \bigcup_{i \in \Gamma} A_i(y) \)

⇒ \( \bigcup_{i \in \Gamma} A_i \) is a fuzzy filter

To prove that \( \bigcup_{i \in \Gamma} A_i \) is a fuzzy b-completely closed filter of \( X \)

Let \( x,y \in F \), \( b \in X \)

\( (\bigcup_{i \in \Gamma} A_i(b^*(x*y))) = \sup\{ Ai(b^*(x*y)), i \in \Gamma \} \)

\geq \sup\{ \min\{ Ai(x),Ai(y)\}, i \in \Gamma \} \)

\geq \min\{ \sup Ai(x),\inf Ai(y), i \in \Gamma \} \)

\geq \min\{ \bigcup_{i \in \Gamma} A_i(x), \bigcup_{i \in \Gamma} A_i(y) \} \)

⇒ \( \bigcup_{i \in \Gamma} A_i(b^*(x*y)) \geq \min\{ (\bigcup_{i \in \Gamma} A_i)(x), (\bigcup_{i \in \Gamma} A_i)(y) \} \quad \forall x,y \in F \)

Therefore \( \bigcup_{i \in \Gamma} A_i \) is a fuzzy b-completely closed filter of \( X \). ■

Proposition (2.10):

Let \( \{Ai:i \in \Gamma\} \) be a family of fuzzy b-closed filter of a BH-algebra \( X \). Then \( \bigcap_{i \in \Gamma} A_i \) is a fuzzy b-closed filter of \( X \).

Proof:

\( \bigcap_{i \in \Gamma} A_i \) is a fuzzy filter \quad \text{[proposition (2.8)]}

\( \bigcap_{i \in \Gamma} A_i \) to prove is a fuzzy b-closed filter

Let \( b \in X \), \( x \in F \)

\( (\bigcap_{i \in \Gamma} A_i)(b^*(0*x)) = \inf\{ Ai(b^*(0*x)), i \in \Gamma \} \)

\geq \inf\{ Ai(x), i \in \Gamma \} \)

\geq (\bigcap_{i \in \Gamma} A_i)(x), \)

⇒ \( (\bigcap_{i \in \Gamma} A_i)(b^*(0*x)) \geq (\bigcap_{i \in \Gamma} A_i)(x) \)

Therefore, \( \bigcap_{i \in \Gamma} A_i \) is a fuzzy b-closed filter of \( X \). ■

Proposition (2.11):

Let \( \{Ai:i \in \Gamma\} \) be a family of fuzzy b-closed filter of a BH-algebra \( X \). Then \( \bigcup_{i \in \Gamma} A_i \) is a fuzzy b-closed filter of \( X \).

Proof:

\( \bigcup_{i \in \Gamma} A_i \) is a fuzzy filter \quad \text{[proposition (2.9)]}

\( \bigcup_{i \in \Gamma} A_i \) to prove is a fuzzy b-closed filter
\[
\left( \bigcup_{i \in \Gamma} A_i \right) (b^*(0^*x)) = \sup \{ A_i(b^*(0^*x)) : i \in \Gamma \} \\
\geq \sup \{ A_i(x) : i \in \Gamma \} \\
\geq \left( \bigcup_{i \in \Gamma} A_i \right) (x)
\]
\[
\Rightarrow \left( \bigcup_{i \in \Gamma} A_i \right) (b^*(0^*x)) \geq \left( \bigcup_{i \in \Gamma} A_i \right) (x)
\]
Therefore, \( \left( \bigcup_{i \in \Gamma} A_i \right) \) is a fuzzy b-closed filter of X. ■

**Proposition (2.12):**

Let X be BH-algebra and A be a fuzzy set of X. Then A is a fuzzy b-completely closed filter if and only if \( A'(x) = A(x) + 1 - A(0) \) is a fuzzy b-completely closed filter.

**Proof:**

Let A be a fuzzy b-completely closed filter,
\[
\Rightarrow A \text{ is a fuzzy filter. } \quad \text{[Proposition (2.3)]}
\]
\[
\Rightarrow A' \text{ is a fuzzy filter. } \quad \text{[By theorem(1.15)]}
\]
Now, Let \( x, y \in A, b \in X \)
\[
A(b^*(x^*y)) = A(b^*(x^*y)) + 1 - A(0) \\
\geq \min \{ A(x), A(y) \} + 1 - A(0) \quad \text{[Since A is a fuzzy b-completely closed filter]} \\
\geq \min \{ A(x) + 1 - A(0), A(y) + 1 - A(0) \} \\
\geq \min \{ A'(x), A'(y) \}
\]
\[
\therefore A'(x) \geq \min \{ A'(x), A'(y) \}
\]
\[
\therefore A' \text{ is a fuzzy b-completely closed filter}
\]

Conversely

Let \( A' \) be a fuzzy b-completely closed filter,
\[
\Rightarrow A' \text{ is a fuzzy filter. } \quad \text{[By theorem(1.15)]}
\]
\[
\Rightarrow A \text{ is a fuzzy filter. } \quad \text{[By theorem(1.15)]}
\]
Now, Let \( x, y \in A, b \in X \)
\[
A(b^*(x^*y)) = A(b^*(x^*y)) - 1 + A(0) \\
\geq \min \{ A'(x), A'(y) \} - 1 + A(0) \quad \text{[Since A' is a fuzzy filter, definition (1.9)]} \\
\geq \min \{ A'(x) - 1 + A(0), A'(y) - 1 + A(0) \} \\
\geq \min \{ A(x), A(y) \}
\]
\[
\therefore A(b^*(x^*y)) \geq \min \{ A(x), A(y) \}
\]
\[
\therefore A \text{ is a fuzzy b-completely closed filter.} ■
\]

**Proposition (2.13):** Let X be BH-algebra and A be a fuzzy set of X. Then every fuzzy completely closed filter is a fuzzy b-completely closed ideal, \( \forall b \in X, A(b) = A(0) \).

**Proof:** Let A be a fuzzy completely closed filter.

To prove A is a fuzzy ideal,

1. \( A(0) \geq A(x) \quad \forall x \in X \quad \text{[Proposition (1.16)]} \)
2. \( A(x) = A(x^0) = A(x^*(y^y)) = A((x^y)^y) \quad \text{[Since X is an associative. By definition(1.5)]} \)
\[ \geq \min \{ A(x*y), A(y) \} \] [Since \( A \) is a fuzzy completely closed filter By definition(1.11)]

.: \( A \) is a fuzzy ideal.

\[ A(b*(x*y)) \geq \min \{ A(b), A(x*y) \} \] [\( A \) is a fuzzy completely closed filter. By definition(1.11)]

\[ \geq \min \{ A(0), A(x*y) \} \]
\[ \geq A(x*y) \]
\[ \geq \min \{ A(x), A(y) \} \]

.: \( A \) is a fuzzy b-completely closed ideal.

**Proposition (2.14):**
Let \( X \) be BH-algebra such that if \( x*y=0 \) implies \( x=y \ \forall x,y \in X \). Then every fuzzy sub algebra is a fuzzy filter.

**Proof:** Let \( A \) be a fuzzy sub algebra and \( x,y \in X \).

1) \( A(x*(x*y)) \geq \min \{ A(x), A(x*y) \} \) [Since \( A \) is a fuzzy sub algebra, definition (1.8)]

\[
\begin{align*}
\text{If } & \min \{ A(x), A(x*y) \} = A(x) \geq \min \{ A(x), A(y) \} \\
\text{If } & \min \{ A(x), A(x*y) \} = A(x*y) \geq \min \{ A(x), A(y) \}
\end{align*}
\]

.: \( A(x*(x*y)) \geq \min \{ A(x), A(y) \} \)

Similarly, \( A(y*(y*x)) \geq \min \{ A(x), A(y) \} \)

2) Let \( x \leq y \)

\[ \Rightarrow x*y=0 \Rightarrow x=y \]

\[ \Rightarrow A(y) = A(x) \geq A(x) \]

.: \( A \) is a fuzzy filter

**Theorem (2.15):**
Let \( X \) be BH-algebra such that \( x*y=0 \) implies \( x=y \ \forall x,y \in X \). Then every fuzzy sub algebra is a fuzzy b-completely closed filter, \( \forall b \in X \) such that \( A(b) = A(0) \).

**Proof:**

\[ A \] is a fuzzy filter [proposition(2.14)]

\[ A(b*(x*y)) \geq \min \{ A(b), A(x*y) \} \] [\( A \) is a fuzzy sub algebra, definition (1.8)]

\[ \geq \min \{ A(0), A(x*y) \} \]
\[ \geq A(x*y) \] [\( A \) is a fuzzy sub algebra, definition (1.8)]
\[ \geq \min \{ A(x), A(y) \} \]

.: \( A \) is a fuzzy b-completely closed filter.

4. REFERENCES


