Kind of Weak Separation Axioms by $D_\omega$, $D_{\alpha-\omega}$, $D_{\text{pre}-\omega}$, $D_{b-\omega}$
and $D_{\beta-\omega}$ – Sets

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Abstract---- In this paper we define new types of sets we call them $D_\omega$, $D_{\alpha-\omega}$, $D_{\text{pre}-\omega}$, $D_{b-\omega}$ and $D_{\beta-\omega}$ – sets and use them to define some associative separation axioms. Some theorems about the relation between them and the weak separation axioms introduced in [5] are proved, with some other simple theorems.

Keywords--- Separation axioms, weak open sets, $T_1$ spaces.

1. Introduction

Throughout this paper, $(X,T)$ stands for topological space. Let $(X,T)$ be a topological space and $A$ a subset of $X$. A point $x$ in $X$ is called condensation point of $A$ if for each $U$ in $T$ with $x$ in $U$, the set $U \cap A$ is uncountable [6]. In 1982 the $\omega$ – closed set was first introduced by H. Z. Hedeb in [6], and he defined it as: $A$ is $\omega$ – closed if it contains all its condensation points and the $\omega$ – open set is the complement of the $\omega$ – closed set. Equivalently, a subset $W$ of a space $(X,T)$, is $\omega$ – open if and only if for each $x \in W$, there exists $U \in T$ such that $x \in U$ and $U \cap W$ is countable. The collection of all $\omega$ – open sets of $(X,T)$ denoted $T_\omega$ form topology on $X$ and it is finer than $T$. Several characterizations of $\omega$ – closed sets were provided in [1,6,7].

In [3,8,9] some authors introduced $\alpha – open$, pre – open, $b$ – open, and $\beta$ – open sets. On the other hand in [10] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced the notions $\alpha – open$, pre – open, $\beta – open$, and $b – open$ sets in topological spaces. In [2,5] used the $\omega – open$ sets to define types of weak separation axioms called $\omega – R_0$, $\omega – R_1$ and $\omega^+ – T_1$ spaces. They defined them as follows:

Definition 1.1. [10] A subset $A$ of a space $X$ is called:

1. $\alpha – \omega$ open if $A \subseteq \text{int}_\omega(\text{cl}(\text{int}_\omega(A)))$ and the complement of the $\alpha – \omega$ open set is called $\alpha – \omega$ – closed set.
2. pre – $\omega$ open if $A \subseteq \text{int}_\omega(\text{cl}(A))$ and the complement of the pre – $\omega$ – open set is called pre – $\omega$ – closed set.
3. $b – \omega$ open if $A \subseteq \text{int}_\omega(\text{cl}(A)) \cup \text{cl}(\text{int}_\omega(A))$ and the complement of the $b – \omega$ – open set is called $b – \omega$ – closed set.
4. $\beta – \omega$ open if $A \subseteq \text{cl}(\text{int}_\omega(\text{cl}(A)))$ and the complement of the $\beta – \omega$ – open set is called $\beta – \omega$ – closed set.

In [10] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced relationships among the weak open sets above by the lemma below:

Lemma 1.2. [10] In any topological space:

1. Any open set is $\omega$ – open.
2. Any $\omega$ – open set is $\alpha$ – $\omega$ – open.
3. Any $\alpha – \omega$ – open set is pre – $\omega$ – open.
4. Any pre – $\omega$ – open set is $b$ – $\omega$ – open.
5. Any $b$ – $\omega$ – open set is $\beta$ – $\omega$ – open.
The converse is not true [10].

For our results in this paper we need the following definitions:

**Definition 1.3.** [10] A subset $A$ of a space $X$ is called
1. An $\omega - t -$ set, if $\operatorname{int}(A) = \operatorname{int}_\omega (\operatorname{cl}(A))$.
2. An $\omega - B -$ set if $A = U \cap V$, where $U$ is an open set and $V$ is an $\omega - t -$ set.
3. An $\omega - t_\alpha -$ set, if $\operatorname{int}(A) = \operatorname{int}_\omega (\operatorname{cl}(\operatorname{int}(A)))$.
4. An $\omega - B_\alpha -$ set if $A = U \cap V$, where $U$ is an open set and $V$ is an $\omega - t_\alpha -$ set.
5. An $\omega -$ set if $A = U \cap V$, where $U$ is an open set and $\operatorname{int}(V) = \operatorname{int}_\omega (V)$.

**Definition 1.4.** [5] Let $(X, T)$ be topological space. It said to be satisfy
1. The $\omega -$condition if every $\omega -$open set is $\omega - t -$set.
2. The $\omega - B_\alpha -$condition if every $\alpha -$open set is $\omega - B_\alpha -$set.
3. The $\omega - B -$condition if every pre-$\omega -$open is $\omega - B -$set.

**Lemma 1.5.** [10] For any subset $A$ of a space $X$, We have
1. A is open if and only if $A$ is $\omega -$open and $\omega -$set.
2. A is open If and only if $A$ is $\alpha -$open and $\omega - B_\alpha -$set.
3. A is open if and only if $A$ is pre-$\omega -$open and $\omega - B -$set.

**Lemma 1.6.** If $(X, T)$ is a door space, then
1. Every pre-$\omega -$open set is $\omega -$open. [10]
2. Every $\beta -$ $\omega -$open set is is $b -$open.[5]

**Lemma 1.7.** [10] Let $(X, T)$ be a topological space and let $A \subseteq X$. If $A$ is $b - \omega -$open set such that $\operatorname{int}_\omega (A) = \emptyset$, then $A$ is pre-$\omega -$open.

The classes of the sets in Definition 1.1 are larger than that sets in [3,8,9]. In [5] we introduce some weak separation axioms by utilizing the notions of T. Noiri, A. Al-Omari, M. S. M. Noorani. Let us summarize them in the following definitions.

**Definition 1.3.**[5] Let $X$ be a topological space. If for each $x \neq y \in X$, either there exists a set $U$, such that $x \in U$, $y \notin U$, or there exists a set $U$ such that $x \notin U$, $y \in U$. Then $X$ called
1. $\omega - T_0$ space, whenever $U$ is $\omega -$open set in $X$.
2. $\alpha - \omega - T_0$ space, whenever $U$ is $\alpha - \omega -$open set in $X$.
3. pre-$\omega - T_0$ space, whenever $U$ is pre-$\omega -$open set in $X$.
4. $b - \omega - T_0$ space, whenever $U$ is $b - \omega -$open set in $X$.
5. $\beta - \omega - T_0$ space, whenever $U$ is $\beta - \omega -$open set in $X$.

**Definition 1.4.**[5] Let $X$ be a topological space. For each $x \neq y \in X$, there exists a set $U$, such that $x \in U$, $y \notin U$, and there exists a set $V$ such that $y \in V$, $x \notin V$, then $X$ called
1. $\omega - T_1$ space if $U$ is open and $V$ is $\omega -$open sets in $X$.
2. $\alpha - \omega - T_1$ space if $U$ is open and $V$ is $\alpha - \omega -$open sets in $X$.
3. $\omega^* - T_1$ space [1] if $U$ and $V$ are $\omega -$open sets in $X$.
4. $\alpha - \omega^* - T_1$ space if $U$ is $\omega$-open and $V$ is $\alpha - \omega -$open sets in $X$.
5. $\alpha - \omega^{**} - T_1$ space if $U$ and $V$ are $\alpha - \omega -$open sets in $X$.
6. pre-$\omega - T_1$ space if $U$ is open and $V$ is pre-$\omega -$open sets in $X$.
7. pre-$\omega^* - T_1$ space if $U$ is $\omega$-open and $V$ is pre-$\omega -$open sets in $X$.
8. $\alpha -$pre-$\omega - T_1$ space if $U$ is $\alpha - \omega -$open and $V$ is pre-$\omega -$open sets in $X$. 

9. pre $-\omega^{**} - T_1$ space if $U$ and $V$ are pre $-\omega$-open sets in $X$.
10. $b - \omega - T_1$ space if $U$ is open and $V$ is $b - \omega$-open sets in $X$.
11. $b - \omega^* - T_1$ space if $U$ is $\omega$-open and $V$ is $b - \omega$-open sets in $X$.
12. $\alpha - b - \omega - T_1$ space if $U$ is $\alpha - \omega$-open and $V$ is $b - \omega$-open sets in $X$.
13. pre $- b - \omega - T_1$ space if $U$ is pre $-\omega$-open and $V$ is $b - \omega$-open sets in $X$.
14. $b - \omega^{**} - T_1$ space if $U$ and $V$ are $b - \omega$-open sets in $X$.
15. $\beta - \omega - T_1$ space if $U$ is open and $V$ is $\beta - \omega$-open sets in $X$.
16. $\beta - \omega^* - T_1$ space if $U$ is $\omega$-open and $V$ is $\beta - \omega$-open sets in $X$.
17. $\alpha - \beta - \omega - T_1$ space if $U$ is $\alpha - \omega$-open and $V$ is $\beta - \omega$-open sets in $X$.
18. $\alpha - \beta - \omega - T_1$ space if $U$ is $\alpha - \omega$-open and $V$ is $\beta - \omega$-open sets in $X$.
19. $\beta - \omega^{**} - T_1$ space if $U$ and $V$ are $\beta - \omega$-open sets in $X$.
20. $b - \beta - \omega - T_1$ space if $U$ is $b - \omega$-open and $V$ is $\beta - \omega$-open sets in $X$.

**Definition 1.5.** [5] Let $X$ be a topological space. And for each $x \neq y \in X$ there exist two disjoint sets $U$ and $V$ with $x \in U$ and $y \in V$, then $X$ is called:
1. $\omega - T_2$ space if $U$ is open and $V$ is $\omega$-open sets in $X$.
2. $\alpha - \omega - T_2$ space if $U$ is open and $V$ is $\alpha - \omega$-open sets in $X$.
3. $\omega^* - T_2$ space if $U$ and $V$ are $\omega$-open sets in $X$.
4. $\alpha - \omega^* - T_2$ space if $U$ is $\omega$-open and $V$ is $\alpha - \omega$-open sets in $X$.
5. $\alpha - \omega^{**} - T_2$ space if $U$ and $V$ are $\alpha - \omega$-open sets in $X$.
6. pre $-\omega - T_2$ space if $U$ is open and $V$ is pre $-\omega$-open sets in $X$.
7. pre $-\omega^* - T_2$ space if $U$ is $\omega$-open and $V$ is pre $-\omega$-open sets in $X$.
8. $\alpha - \omega - T_2$ space if $U$ is $\alpha$-open and $V$ is $\omega$-open sets in $X$.
9. pre $-\omega^{**} - T_2$ space if $U$ and $V$ are pre $-\omega$-open sets in $X$.
10. $b - \omega - T_2$ space if $U$ is open and $V$ is $b - \omega$-open sets in $X$.
11. $b - \omega^* - T_2$ space if $U$ is $\omega$-open and $V$ is $b - \omega$-open sets in $X$.
12. $\alpha - b - \omega - T_2$ space if $U$ is $\alpha - \omega$-open and $V$ is $b - \omega$-open sets in $X$.
13. pre $- b - \omega - T_2$ space if $U$ is pre $-\omega$-open and $V$ is $b - \omega$-open sets in $X$.
14. $b - \omega^{**} - T_2$ space if $U$ and $V$ are $b - \omega$-open sets in $X$.
15. $\beta - \omega - T_2$ space if $U$ is open and $V$ is $\beta - \omega$-open sets in $X$.
16. $\beta - \omega^* - T_2$ space if $U$ is $\omega$-open and $V$ is $\beta - \omega$-open sets in $X$.
17. $\alpha - \beta - \omega - T_2$ space if $U$ is $\alpha - \omega$-open and $V$ is $\beta - \omega$-open sets in $X$.
18. pre$-\beta - \omega - T_2$ space if $U$ is pre $-\omega$-open and $V$ is $\beta - \omega$-open sets in $X$.
19. $\beta - \omega^{**} - T_2$ space if $U$ and $V$ are $\beta - \omega$-open sets in $X$.
20. $b - \beta - \omega - T_2$ space if $U$ is $b - \omega$-open and $V$ is $\beta - \omega$-open sets in $X$.

2. $D_\omega$, $D_{\alpha - \omega}$, $D_{\text{pre} - \omega}$, $D_{b - \omega}$ AND $D_{\beta - \omega}$ --SETS

In this article we shall define new types of sets and use them to define new spaces with associative separation axioms.
**Definition 2.1.** A subset $A$ of a topological space $(X, T)$ is called $D$–set [4] (resp. $D_{\omega}$–set, $D_{\alpha-\omega}$–set, $D_{\text{pre}-\omega}$–set, $D_{b-\omega}$–set, $D_{\beta-\omega}$–set). If there are two open (resp. $\omega$–open, $\alpha$–$\omega$–open, $\text{pre}$–$\omega$–open, $\beta$–$\omega$–open, and $b$–$\omega$–open) sets $U$ and $V$ with $U \neq X$ and $A = U \setminus V$.

**Remark 2.2.** It is true that every $\omega$–open, (resp. $\alpha$–$\omega$–open, $\text{pre}$–$\omega$–open, $b$–$\omega$–open, and $\beta$–$\omega$–open) set $U \neq X$ is $D_{\omega}$–set (resp. $D_{\alpha-\omega}$–set, $D_{\text{pre}-\omega}$–set, $D_{b-\omega}$–set, and $D_{\beta-\omega}$–set) if $A = U$ and $V = \emptyset$.

Using Definition 2.1 and Lemma 1.2, Lemma 1.6, and Lemma 1.5 we can easily prove the following Propositions:

**Proposition 2.3.** In any topological space $X$.
1. Any $D$–set is $D_{\omega}$–set.
2. Any $D_{\alpha}$–set is $D_{\alpha-\omega}$–set.
3. Any $D_{\alpha-\omega}$–set is $D_{\text{pre}-\omega}$–set.
4. Any $D_{\text{pre}-\omega}$–set is $D_{b-\omega}$–set.
5. Any $D_{b-\omega}$–set is $D_{\beta-\omega}$–set.

**Proposition 2.4.** In any topological door space:
1. Any $D_{\text{pre}-\omega}$–set is $D_{\omega}$–set.
2. Any $D_{\beta-\omega}$–set is $D_{b-\omega}$–set.

**Proposition 2.5.** In any topological space satisfies $\omega$–condition. Any $D_{\omega}$–set is $D$–set.

**Proposition 2.6.** In any topological space satisfies $\omega$–$\alpha$–condition. Any $D_{\alpha-\omega}$–set is $D$–set.

**Proposition 2.7.** In any topological space satisfies $\omega$–$\beta$–condition. Any $D_{\text{pre}-\omega}$–set is $D$–set.

**Proposition 2.8.** In any topological space. Any $D_{b-\omega}$–set with empty $\omega$–interior is $D_{\text{pre}-\omega}$–set.

**Proof:**
Let $X$ be a topological space, and let $A$ be a $D_{b-\omega}$–set with empty $\omega$–interior in $X$, then there are two $b$–$\omega$–open which are by Lemma 1.7 also $\text{pre}$–$\omega$–open sets $U$ and $V$ with $U \neq X$, and $A = U \setminus V$. Similarly we can prove the other cases.

From the lemmas above we can get the following figure:

![Figure 1: Relation among the weak D–sets](image-url)
Utilizing the weak $D_ω$ sets we can define our separation axioms as follows:

**Definition 3.1.** Let $X$ be a topological space. If $x \neq y \in X$, either there exists a set $U$, such that $x \in U$, $y \in U$, or there exists a set $U$ such that $x \notin U$, $y \notin U$. Then $X$ called

1. $ω - D₀$ space, whenever $U$ is $D_ω$ - set in $X$.
2. $α - ω - D₀$ space, whenever $U$ is $D_{α-ω}$ - set in $X$.
3. $pre-ω - D₀$ space, whenever $U$ is $D_{pre-ω}$ - set in $X$.
4. $b - ω - D₀$ space, whenever $U$ is $D_{b-ω}$ - set in $X$.
5. $β - ω - D₀$ space, whenever $U$ is $D_{β-ω}$ - set in $X$.

**Definition 3.2.** We can define the spaces $ω - D₁$, $α - ω - D₁$, $α - ω^* - D₁$, $pre - ω - D₁$, $b - ω - D₁$, $β - ω - D₁$, for $i = 0, 1, 2$. And $ω^* - D₁$, $α - ω^* - D₁$, $α - ω** - D₁$, $pre - ω^* - D₁$, $α - ω** - D₁$, $pre - ω** - D₁$, $b - ω^* - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, and $b - β - ω - D₁$, for $i = 1, 2$, by replacing the sets: open, $α$ - open, $α$ - open, $β$ - open, open, $β$ - open, $β$ - open, by the $D$ - set, $D_{α-ω}$ - set, $D_{β-ω}$ - set, $D_{pre-ω}$ - set, $D_{b-ω}$ - set, and $D_{β-ω}$ - set respectively, in Definition 1.3, Definition 1.4, and Definition 1.5.

**Remark 3.3.** For the relations among weak $T_5$s we can make a figures coincide with these for weak $T_5$s spaces in [5].

**Theorem 3.4.** Let $(X, T)$ be a topological space:

1. If $(X, T)$ is $ω - T₀$, (resp. $α - ω - T₀$, $pre - ω - T₀$, $b - ω - T₀$, $β - ω - T₀$), for $i = 0, 1, 2$, then $ω^* - T₀$, $α - ω^* - T₀$, $α - ω** - T₀$, $pre - ω^* - T₀$, $b - ω^* - T₀$, $β - ω** - T₀$, $β - ω** - T₀$, $α - β - ω - T₀$, $β - ω** - T₀$, $α - β - ω - T₀$, and $b - β - ω - T₀$, for $i = 1, 2$, then it is $ω - D₁$, (resp. $α - ω - D₁$, $pre - ω - D₁$, $b - ω - D₁$, $β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, and $b - β - ω - D₁$, for $i = 1, 2$).

2. If $(X, T)$ is $ω - D₁$, (resp. $α - ω - D₁$, $ω^* - D₁$, $α - ω^* - D₁$, $α - ω** - D₁$, $pre - ω - D₁$, $pre - ω^* - D₁$, $pre - ω** - D₁$, $b - ω^* - D₁$, $β - ω^* - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, and $b - β - ω - D₁$, for $i = 1, 2$).

**Proof:**

1. Follows immediately by the Remark 3.3.
2. Directly from Definition 2.1, Definition 3.1, and Definition 3.2.

By the following theorems we recognize the importance of the weak $D_1$-spaces, for $i = 0, 1, 2$.

**Theorem 3.5.** Let $(X, T)$ be a topological space. Then $X$ is $ω - D₁$, (resp. $α - ω - D₁$, $ω^* - D₁$, $α - ω^* - D₁$, $α - ω** - D₁$, $pre - ω - D₁$, $pre - ω^* - D₁$, $pre - ω** - D₁$, $b - ω - D₁$, $β - ω - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, $α - β - ω - D₁$, $β - ω** - D₁$, $β - ω** - D₁$, and $b - β - ω - D₁$, for $i = 1, 2$).

**Proof:**

The proof of the forward direction is a step by step similar to that of Theorem 4.8 in [4]. The inverse direction follows immediately from (2) of theorem 3.4 above.

**Theorem 3.6.** Let $(X, T)$, be a topological space. Then $X$ is $α - ω - T₀$, (resp. $ω - T₀$, $pre - ω - T₀$, $b - ω - T₀$, $β - ω - T₀$) if and only if it is $α - ω - D₀$ (resp. $ω - D₀$, $pre - ω - D₀$, $b - ω - D₀$, $β - ω - D₀$).
Proof:

The forward direction follows immediately from (1) of Theorem 3.4. For the opposite side let $X$ be $\alpha - \omega - D_0$, so for $x \neq y \in X$, there is a $D_{\alpha - \omega} -$ set $U$ such that $x \in U$, but $y \notin U$. Then by the definition of the $D_{\alpha - \omega} -$ set, $U = V \setminus W$, where $V$ and $W \neq X$ are $\alpha - \omega -$ open sets. Now if $x \in W$, but $y \notin W$, and $W$ is an $\alpha - \omega -$ open set in $X$. So $X$ is $\alpha - \omega - T_0$. Then whenever $x \in U = W \setminus V$ and $y \in (W \cap V)$. Then $y \notin V$, and $x \notin V$. Thus $X$ is $\alpha - \omega - T_0$ space.

For the following definition we need the definition of the $\omega -$neighbourhood from [5]:

Definition 3.7. [5] Let $(X, T)$ be a topological space. A subset $U$ of $X$ is $\omega -$neighbourhood of a point $x \in X$, if and only if there exists an $\omega -$open set $V$ such that $x \in V \subseteq U$.

Definition 3.8. A point $x \in X$ which has only $\alpha -$neighbourhood is called an $\alpha -$net point.

Proposition 3.9. Let $(X, T)$ be a topological space If $X$ is $\omega - D_1$ space, then it has no $\omega -$net point.

Proof:

Since $X$ is $\omega - D_1$ so each point $x$ of $X$ contained in a $D_{\omega} -$ set $W = U \setminus V$, $U \neq X$, and $U$ and $V$ are $\omega -$open sets. So it contained in the $\omega -$open set $U \neq X$, which implies $x$ is no $\omega -$net point.

Theorem 3.10. Let $X$ be a door topological space, has no $\omega -$net point. Then it is $\omega - D_1$ space.

Proof:

Since $(X, T)$ be a door topological space, so for each point $x$ in $X$, $(x)$ is either $\omega -$open or $\omega -$closed. This implies for each $x \neq y \in X$, at least one of them say $x$ has $\omega -$neighbourhood $U \neq X$ containing $x$ but not $y$, $U$ is $D_{\omega} -$ set. If $X$ has no $\omega -$net point, then $y$ is not $\omega -$net point, so there is an $\omega -$neighbourhood $V \neq X$ of $y$. Thus $V \setminus U$ is $D_{\omega} -$ set containing $y$ but not $x$. Hence $X$ is $\omega - D_1$ space.

To introduce Theorem 3.12 we need the following Definition from [5]:

Definition 3.11. [5] Let $(X, \sigma)$ and $(Y, \tau)$ be two topological spaces. A map $f: (X, \sigma) \rightarrow (Y, \tau)$ is called $\omega$-continuous (resp. $\alpha - \omega$-continuous, $\alpha - \omega -$precontinuous, $\beta - \omega$-continuous ) at $x \in X$, if and only if for each $\omega -$open (resp. $\alpha - \omega$-open, $\alpha - \omega -$preopen, $\beta - \omega$-open ) set $V$ containing $f(x)$, there exists an $\omega -$open (resp. $\alpha - \omega$-open, $\alpha - \omega -$preopen, $\beta - \omega$-open ) set $U$ containing $x$, such that $f(U) \subseteq V$. If $f$ is $\omega -$ continuous (resp. $\alpha - \omega$-continuous, $\alpha - \omega -$precontinuous, $\beta - \omega$-continuous ) at each $x \in X$, we call it $\omega$-continuous (resp. $\alpha - \omega$-continuous, $\alpha - \omega -$precontinuous, $\beta - \omega$-continuous).

Theorem 3.12. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\omega$-continuous (resp. $\alpha - \omega$-continuous , $\alpha - \omega$-precontinuous, $\beta - \omega$-continuous, $\beta - \omega$-precontinuous ) onto function and $A$ is $D_{\omega} -$ set (resp. $D_{\alpha - \omega} -$ set, $D_{\alpha - \omega} -$ set, $D_{\beta - \omega} -$ set ) in $Y$, then $f^{-1}(A)$ is also $D_{\omega} -$ set (resp. $D_{\alpha - \omega} -$ set, $D_{\alpha - \omega} -$ set, $D_{\beta - \omega} -$ set ) in $X$.

Proof:

Let $A$ be $D_{\omega} -$ set in $Y$, so there are two $\omega -$open sets $U \neq Y, V$ in $Y$ such that $A = U \setminus V$. Then by the $\omega$-continuous function definition, we have $f^{-1}(U) = f^{-1}(V)$ are $\omega -$open sets in $X$, such that $f^{-1}(U) \neq X$. And $f^{-1}(A) = f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$ is $D_{\omega} -$ set in $X$.

The other cases are the same.

Theorem 3.13. For any two topological spaces $(X, \tau)$ and $(Y, \sigma)$.

1. If $(Y, \sigma)$ be an $\alpha^* - D_1$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ is an $\omega$-continuous bijection, then $(X, \tau)$ is $\alpha^* - D_1$.
2. If $(Y, \sigma)$ be an, $\alpha - \omega^*$ - $D_1$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ is an $\alpha - \omega$-continuous bijection, then $(X, \tau)$ is $\alpha - \omega^* - D_1$.
3. If $(Y, \sigma)$ be an, $\alpha - \omega^*$ - $D_1$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ is a $\alpha - \omega$-continuous bijection, then $(X, \tau)$ is $\alpha - \omega^* - D_1$.
4. If $(Y, \sigma)$ be an, $\beta - \omega^*$ - $D_1$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ is a $\beta - \omega$-continuous bijection, then $(X, \tau)$ is $\beta - \omega^* - D_1$.
5. If $(Y, \sigma)$ be an, $\beta - \omega^*$ - $D_1$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ is a $\beta - \omega$-continuous bijection, then $(X, \tau)$ is $\beta - \omega^* - D_1$.

Proof of (1):
Let $Y$ be an $\omega^* - D_1$ space. Let $x \neq y \in X$, since $f$ is bijective and $Y$ is $\omega^* - D_1$ space, so there exist two $D_\omega$ - sets $U$ and $V$ such that $U$ containing $f(x)$ but not $f(y)$ and $V$ containing $f(y)$ but not $f(x)$, then by Theorem 3.12, $f^{-1}(U)$ and $f^{-1}(V)$ are $D_\omega$ - sets such that $f^{-1}(U)$ containing $x$ but not $y$ and $f^{-1}(V)$ containing $y$ but not $x$. So $(X, \tau)$ is $\omega^* - D_1$.

By the same way we can prove the other cases.

Theorem 3.14. A topological space $(X, \tau)$ is $\omega^* - D_1$ ( resp. $\alpha - \omega^* - D_1$, $\beta - \omega^* - D_1$ ) if and only if only for each pair of distinct points $x, y \in X$, there exists an $\omega$ - continuous ( resp. $\alpha$ - $\omega$ - continuous, $\beta$ - $\omega$ - continuous, $\beta - \omega$ - $\omega$ - continuous ) onto function $f : (X, \tau) \to (Y, \sigma)$ such that $f(x)$ and $f(y)$ are distinct, where $(Y, \sigma)$ is $\omega^* - D_1$ ( resp. $\alpha - \omega^* - D_1$, $\beta - \omega^* - D_1$ ) space.

Proof:

Let $(X, \tau)$ be an $\omega^* - D_1$, let $x, y \in X$, then we can find an onto function $f : (X, \tau) \to (Y, \sigma)$, where $(Y, \sigma)$ is an $\omega^* - D_1$ space defined by $f(x) = x$, such that $f(x)$ and $f(y)$ distinct. For the opposite direction. Let $x \neq y \in X$, and $f : (X, \tau) \to (Y, \sigma)$ be an onto $\omega$ - continuous function such that $f(x)$ and $f(y)$ distinct, and $(Y, \sigma)$ is $\omega^* - D_1$ space. We must prove $(X, \tau)$ is $\omega^* - D_1$ space. Since $(Y, \sigma)$ is an $\omega^* - D_1$ space and $f(x)$ and $f(y)$ are distinct points in it, then by Theorem 3.5 there are two distinct disjoint $D_\omega$ - sets $U$ and $V$ in $Y$ such that $U$ containing $f(x)$ and $V$ containing $f(y)$. Then since $f$ is $\omega$ - continuous function so $f^{-1}(U)$ and $f^{-1}(V)$ are two disjoint $D_\omega$ - sets in $X$ such that $f^{-1}(U)$ containing $x$ and $f^{-1}(V)$ containing $y$. So $(X, \tau)$ is $\omega^* - D_1$, and by Theorem 3.5. again , we get $(X, \tau)$ is $\omega^* - D_1$ space.

4. REFERENCES


