The Properties of Generalized k-Pell like Sequence using Matrices

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ABSTRACT— The Pell sequence has been generalized in many ways. In this study, we define new generalization $\{M_{k,n}\}$ with initial conditions $M_{k,0} = 4$, $M_{k,1} = m + 4$, which is generated by the recurrence relation $M_{k,n+1} = kM_{k,n} + M_{k,n-1}$ for $n \geq 1$, where $k, m$ are integer numbers. Then, we obtain some properties related to new generalization of Pell sequence.

Keywords— Pell sequence, recurrence relation

1. INTRODUCTION

The well-known Pell $\{P_n\}$ and Pell-Lucas $\{Q_n\}$ sequences have many interesting properties and their applications to every fields of positive science and art [1-2]. They are defined for $n \geq 2$ with the recurrences $P_n = 2P_{n-1} + P_{n-2}$, ($P_0 = 0$, $P_1 = 1$) and $Q_n = 2Q_{n-1} + Q_{n-2}$, ($Q_0 = 2$, $Q_1 = 2$) respectively. In the literature, these numbers have been generalized in many ways [1-5]. Falcon and Plaza, in [6], defined the $k$-Fibonacci sequence $\{F_{k,n}\}$, $k \geq 1$, $n \geq 1$ and $k$-Lucas sequence $\{L_{k,n}\}$, $k \geq 1$, $n \geq 1$,

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad (F_{k,0} = 0, \ F_{k,1} = 1)$$

and

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \quad (L_{k,0} = 2, \ L_{k,1} = k)$$

respectively. Many properties of these numbers were deduced directly from elementary matrix algebra. Furthermore the 3-dimensional $k$-Fibonacci spirals were studied from a geometric points of view. In [3-4], Taskara N., Uslu K., Gulec H. H., gave the binomial properties Fibonacci and Lucas sequences and obtained some new algebraic results of these numbers. In [2], Horadam showed that some properties involving Pell numbers. Horadam gave Simpson formula

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n$$

for the Pell numbers. In [7], Godase A. D. defined generalized $k$-Fibonacci like sequence using matrices, studied some properties of these numbers.
2. THE GENERALIZED $k$ - PELL LIKE SEQUENCE

By using [7], we defined a new generalization of the $k$ -Pell sequences and gave few terms of this sequence.

**Definition 2.1.** For any integer number $k \geq 1$ and $m \geq 0$ the generalized $k$ -Pell like sequence $M_{k,n}$ is defined by

$$M_{k,n+1} = 2M_{k,n} + kM_{k,n-1}, \quad (n \geq 1), \quad (M_{k,0} = 4, \ M_{k,1} = m + 4).$$

Characteristics equation of the initial recurrence relation is $r^2 - 2r - k = 0$, and characteristics roots are

$$r_1 = 1 + \sqrt{1 + k}, \quad r_2 = 1 - \sqrt{1 + k}.$$  

Characteristics roots verify the properties

$$r_1 - r_2 = 2\sqrt{1 + k}, \quad r_1 + r_2 = 2, \quad r_1r_2 = -k.$$  

It is clear from the definition of the generalized $k$ -Pell like sequence it satisfy

$$M_{k,n} = mP_{k,n} + Q_{k,n}, \quad (n \geq 0) \quad (2.1)$$

where $P_{k,n}$ and $Q_{k,n}$ are $k$ -Pell and $k$ -Pell-Lucas numbers respectively. $P_{k,n}$ and $Q_{k,n}$ are defined by the solutions of the following discrete equalities

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \quad (n \geq 1)$$

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \quad (n \geq 1)$$

with initial conditions $P_{k,0} = 0$, $P_{k,1} = 1$ and $Q_{k,0} = 2$, $Q_{k,1} = 2$, respectively.

**First few terms of the generalized $k$ -Pell like sequences are:**

$$M_{k,0} = 4,$$

$$M_{k,1} = m + 4,$$

$$M_{k,2} = 4k + 2m + 8,$$

$$M_{k,3} = (m + 12)k + 4m + 16,$$

$$M_{k,4} = 4k^2 + (4m + 32)k + 8m + 32,$$

$$M_{k,5} = (20 + m)k^2 + (12m + 80)k + 16m + 64,$$

$$M_{k,6} = 4k^3 + (6m + 72)k^2 + (32m + 192)k + 32m + 128.$$
3. PROPERTIES OF GENERALIZED $k$ - PELL LIKE SEQUENCE BY MATRIX METHODS

In this section we give our obtained results related to k-Pell Like sequence.

Theorem 3.1. For the generalized $k$ -Pell like sequence $M_{k,n}$ , the follows equality holds

$$
\begin{pmatrix}
M_{k,n+1} & M_{k,n} \\
M_{k,n} & M_{k,n-1}
\end{pmatrix}
= L^n
\begin{pmatrix}
m + 4 & 4 \\
4 & (m - 4)/k
\end{pmatrix},
$$

where $L = \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix}$.  \hfill (3.1)

Proof: Let us use the principle of mathematical induction on $n$. For $n = 1$, it is easy to see that the equality holds

$$
\begin{pmatrix}
M_{k,2} & M_{k,1} \\
M_{k,1} & M_{k,0}
\end{pmatrix}
= \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix}
\begin{pmatrix}
m + 4 & 4 \\
4 & (m - 4)/k
\end{pmatrix}
= \begin{pmatrix} 4k + 2m + 8 & m + 4 \\ m + 4 & 4 \end{pmatrix}.
$$

Now, assume that result is true for $n - 1$. Therefore we have

$$
\begin{pmatrix}
M_{k,n} & M_{k,n-1} \\
M_{k,n-1} & M_{k,n-2}
\end{pmatrix}
= L^{n-1}
\begin{pmatrix}
m + 4 & 4 \\
4 & (m - 4)/k
\end{pmatrix}.
$$

Now, if we multiply the matrix $L$ the last equation, then we can write the following equation

$$
\begin{pmatrix}
M_{k,n} & M_{k,n-1} \\
M_{k,n-1} & M_{k,n-2}
\end{pmatrix}
\begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix}
= L^{n-1}
\begin{pmatrix}
m + 4 & 4 \\
4 & (m - 4)/k
\end{pmatrix}
\begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix},
$$

$$
\begin{pmatrix}
M_{k,n+1} & M_{k,n} \\
M_{k,n} & M_{k,n-1}
\end{pmatrix}
= L^n
\begin{pmatrix}
m + 4 & 4 \\
4 & (m - 4)/k
\end{pmatrix}.
$$

Theorem 3.2. (Simpson’s identity for negative $n$)

$$
M_{k,-n+1}M_{k,-n-1} - M_{k,-n}^2 = \frac{m^2 - 16k - 16}{k}
$$

Proof: If we get $-n$ instead of $n$ in matrix equation 3.1., then we have

$$
\begin{pmatrix}
M_{k,-n+1} & M_{k,-n} \\
M_{k,-n} & M_{k,-n-1}
\end{pmatrix}
= L^{-n}
\begin{pmatrix}
m + 4 & 4 \\
4 & (m - 4)/k
\end{pmatrix}.
\[
L^n = \begin{pmatrix}
P_{k,n+1} & kP_{k,n} \\
P_{k,n} & kP_{k,n-1}
\end{pmatrix} = \frac{1}{k^n \left( P_{k,n+1} P_{k,n-1} - P_{k,n}^2 \right)} \begin{pmatrix}
kP_{k,n+1} & -kP_{k,n} \\
-P_{k,n} & P_{k,n-1}
\end{pmatrix} = \frac{1}{(-1)^n k^n} \begin{pmatrix}
kP_{k,n+1} & -kP_{k,n} \\
-P_{k,n} & P_{k,n+1}
\end{pmatrix}
\]

From the last equations, we can write

\[
\begin{pmatrix}
M_{k,-n+1} & M_{k,-n} \\
M_{k,-n} & M_{k,-n-1}
\end{pmatrix} = \frac{1}{(-1)^n k^n} \begin{pmatrix}
kP_{k,n-1} & -kP_{k,n} \\
-P_{k,n} & P_{k,n+1}
\end{pmatrix} \begin{pmatrix}
m+4 & 4 \\
4 & (m-4)/k
\end{pmatrix}.
\]

If we calculate the determinant of above matrix equation, we have

\[
M_{k,-n+1} M_{k,-n-1} - M_{k,-n}^2 = \frac{1}{(-1)^n k^n} \left[ P_{k,n+1} P_{k,n-1} - P_{k,n}^2 \right] \left[ (m+4)(m-4) - 16k \right]
\]

\[
M_{k,-n+1} M_{k,-n-1} - M_{k,-n}^2 = \frac{1}{(-1)^n k^n} \left[ k^{n-1} (-1)^n \right] \left[ m^2 - 16k - 16 \right] = \frac{m^2 - 16k - 16}{k}.
\]

**Theorem 3.** For arbitrary integer \( n, r \geq 0 \), we have following equalities

\[
M_{k,n+r+1} = (-1)^r (k)^{r+1} \left[ M_{k,n+1} P_{k,r} - M_{k,n} P_{k,r+1} \right],
\]

\[
M_{k,n+r} = (-1)^r (k)^{r+1} \left[ M_{k,n} P_{k,r} - M_{k,n-1} P_{k,r+1} \right],
\]

\[
M_{k,n-r} = (-1)^r (k)^{-r} \left[ M_{k,n-1} P_{k,r} - M_{k,n} P_{k,r+1} \right].
\]

**Proof:** It is obvious

\[
L^{n-r} = \frac{1}{(-1)^r k^r} \begin{pmatrix}
kP_{k,r-1} & -kP_{k,r} \\
-P_{k,r} & P_{k,r+1}
\end{pmatrix} L^n
\]

and

\[
\begin{pmatrix}
M_{k,n+r+1} & M_{k,n+r} \\
M_{k,n+r} & M_{k,n+r-1}
\end{pmatrix} = L^{-r} \begin{pmatrix}
m+4 & 4 \\
4 & (m-4)/k
\end{pmatrix}.
\] (3.3.1)

Otherwise we can write,
From the (3.3.1) and (3.3.2), we have

\[
\begin{pmatrix}
M_{k,n-r+1} & M_{k,n-r} \\
M_{k,n-r} & M_{k,n-r-1}
\end{pmatrix} = \frac{1}{(-1)^r k^r} \begin{pmatrix}
kP_{k,r-1} & -kP_{k,r} \\
-P_{k,r} & P_{k,r+1}
\end{pmatrix} \begin{pmatrix}
m+4 & 4 \\
4 & (m-4)/k
\end{pmatrix}.
\]  

(3.3.2)

Then we have the following results from the above matrix equality

\[
M_{k,n-r+1} = (-1)^r (k)^{1-r} \left[ M_{k,n-r} P_{k,r-1} - M_{k,n} P_{k,r} \right],
\]

\[
M_{k,n-r} = (-1)^r (k)^{-r} \left[ M_{k,n-r} P_{k,r+1} - M_{k,n-1} P_{k,r} \right],
\]

\[
M_{k,n-r-1} = (-1)^r (k)^{-r} \left[ M_{k,n-1} P_{k,r+1} - M_{k,n} P_{k,r} \right].
\]

4. REFERENCES