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A Note on $\widetilde{\mathcal{J}}_E$ - Simple Left Restriction ω −Semigroup

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ABSTRACT---- ***In this paper, we study the $\widetilde{\mathcal{J}}_E$ -simple left restriction ω- semigroup using the Bruck-Reilly extension $BR(M,\theta)$ of a monoid M determined by a morphism θ. In particular, we characterize the Green's ~ -relations in $BR(M,\theta)$. Consequently, we prove that $BR(M,\theta)$ is a $\widetilde{\mathcal{J}}_E$ -simple left restriction ω- semigroup.***

Keywords*:* $\widetilde{\mathcal{J}}_E$ -simple semigroup, left restriction, Green's ~ - relations, ω- semigroup

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1. INTRODUCTION

As noted by Howie [6], the Bruck-Reilly extension $\mathrm{BR}(\mathrm{M},\theta)$ of a monoid M determined by a morphism θ completely defines a bisimple inverse ω-semigroup. Earlier results by Asibong-Ibe [1] described *-bisimple ample ω- semigroup as a kind of Bruck-Reilly extension over a cancellative monoid. Asibong-Ibe [2] proved that a similar result in [1] holds for *-simple ample ω- semigroup. Further investigation by Yu Shang and Limin Wang [7] also described *-bisimple ample I-semigroup as a generalized Bruck-Reilly extension of a monoid determined by a morphism.

Gould [5] introduced a wider class of inverse semigroups which stems from the left ample semigroup of Fountain [3] via the route of replacing the relations $\mathcal{R}^*$ in a semigroup S by those of $\widetilde{\mathcal{R}}_E$ (making reference to a specific set of idempotent E, which may not be the whole of the idempotents $E(S)$). This class of semigroup is known as left restriction semigroup. Now since $\mathrm{BR}(\mathrm{M},\theta)$ defines the *- bisimple ample ω-semigroup and *-simple ample ω-semigroup, it is natural to ask whether $\mathrm{BR}(\mathrm{M},\theta)$ also defines the left restriction semigroup. In this paper, we focus on showing that $\mathrm{BR}(\mathrm{M},\theta)$ is a $\widetilde{\mathcal{J}}_E$ -simple left restriction ω- semigroup.

2. PRELIMINARIES

In this section we recall some definitions as well as some known results which will be useful in the sequel.

Definition 2.1. Let S be a semigroup and let $E \subseteq E(S)$ (E is the distinguished semilattice of idempotents).
Let $a, b \in S$, we have following relations on S

$$a\widetilde{\mathcal{R}}_E b \Leftrightarrow \forall\ e \in E,\ \ ea = a \Leftrightarrow eb = b$$
$$a\widetilde{\mathcal{L}}_E b \Leftrightarrow \forall\ e \in E,\ \ ae = a \Leftrightarrow be = b$$
$$a\widetilde{\mathcal{D}}_E b \Leftrightarrow \exists\, c \in S \text{ such that } a\widetilde{\mathcal{L}}_E c\widetilde{\mathcal{R}}_E b \text{ that is } \widetilde{\mathcal{D}}_E = \widetilde{\mathcal{L}}_E \vee \widetilde{\mathcal{R}}_E.$$

Definition 2.2. Let S be a semigroup. Then S is said to be left (right) ample if
i) every element $a \in S$ is $\mathcal{R}^*(\mathcal{L}^*)$ − related to an idempotent, denoted by $a^\dagger$ (a^*)
ii) for all $a \in S$ and all $e \in E(S)$,

$$ae = (ae)^\dagger a \qquad (\ ea = a(ea)^*\ ).$$

Definition 2.3. Let S be a semigroup and let $E \subseteq E(S)$. Then S is said to be left (right) restriction semigroup if
i) E is a semilattice
ii) every element $a \in S$ is $\widetilde{\mathcal{R}}_E$ $(\widetilde{\mathcal{L}}_E)$– related to an idempotent of E, denoted by $a^\dagger$ (a^*)
iii) the relation $\widetilde{\mathcal{R}}_E$ $(\widetilde{\mathcal{L}}_E)$ is a left (right) congruence
iv) the left (right) ample condition holds:

$$ae = (ae)^\dagger a \qquad (\ ea = a(ea)^*\ ).$$

Definition 2.4. Let S be a left (right) restriction semigroup. A left (right) ideal I of S is said to be a ~ - left (right) ideal if it is the union of $\widetilde{\mathcal{R}}_E$ $(\widetilde{\mathcal{L}}_E)$ -classes, that is, if $a \in I$ then $\widetilde{R}_a$ $(\widetilde{L}_a) \subseteq I$. The smallest ~ - left (right) ideal containing a which is the union of $\widetilde{\mathcal{D}}_E$ -classes is denoted by $\widetilde{J}(a)$. We define the relation $\widetilde{\mathcal{J}}_E$ on S by $a\widetilde{\mathcal{J}}_E b \Leftrightarrow \widetilde{J}(a) = \widetilde{J}(b)$. A left (right) restriction semigroup S is said to be $\widetilde{\mathcal{J}}_E$-simple if $\widetilde{\mathcal{J}}_E$ is the universal relation.

Lemma 2.5 [4]. Let S be a semigroup and $a, b \in S$. Then $b \in \widetilde{J}(b)$ if and only if there are elements $a_0, a_1, \dots, a_n \in S, x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in S^1$ such that $a = a_0, b = a_n$ and $a_i\widetilde{\mathcal{D}}_E\, x_i a_{i-1} y_i$, for $i = 1, 2, \dots, n$.

Lemma 2.6 [3]. Let S be a semigroup and e be an idempotent in S. Then the following are equivalent for $a \in S$.
i) $a\,\mathcal{R}^* e$
ii) $ea = a$, and for all $x, y \in S^1$, $xa = ya$ implies $xe = ye$.

Lemma 2.7 [5]. Let S be a semigroup and $E \subseteq E(S)$, let $a \in S, e \in E$. Then the following conditions are equivalent:

i) $a\, \tilde{\mathcal{R}}_E\, e$
ii) $ea = a$ and for all $f \in E, fa = a \Rightarrow fe = e$.

In a similar way to the *-relations, the ~ -relations are also related to the Green's relations as follows:

Lemma 2.8 [5]. In any semigroup S we have $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}}_E$. If S is regular, and $E = E(S)$ then $\tilde{\mathcal{R}}_E \subseteq \mathcal{R}$ and so $\tilde{\mathcal{R}}_E \subseteq \mathcal{R}^*$.

Dually we have $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \tilde{\mathcal{L}}_E$, and if S is regular, and $E = E(S)$ then $\tilde{\mathcal{L}}_E \subseteq \mathcal{L}$ and so $\tilde{\mathcal{L}}_E \subseteq \mathcal{L}^*$.

3. BRUCK-REILLY EXTENSION OF A MONOID

Let M be a monoid with identity e and $\theta : M \to M$ be a morphism. Let θ^0 be the identity map on M and $S = BR(M, \theta)$ consist of set $S = \mathbb{N}^0 \times M \times \mathbb{N}^0$ (where $\mathbb{N}^0$ denote the set of non-negative integers) with multiplication defined by the rule

$$(m, x, n)(p, y, q) = (m - n + t, x\theta^{t-n} y\theta^{t-p}, q - p + t)$$

where $t = \max(n, p)$, for $(m, x, n), (p, y, q) \in S$.
This construction is a generalization of constructions by Bruck and Reilly, thus $BR(M, \theta)$ is known as the Bruck-Reilly extension of a monoid determined by morphism.

Proposition 3.1 [6]. $BR(M, \theta)$ is a semigroup.
It is also important to note that the idempotents of $BR(M, \theta)$ are of the form (m, e, m) where $m \in \mathbb{N}^0$ and $e \in E(M)$.
Proposition 3.2 [6]. $BR(M, \theta)$ is regular if and only if M is regular.
From Proposition 3.2 [6], we know that $BR(M, \theta)$ is an inverse semigroup.
Proposition 3.3 [6]. Let $(m, x, n), (p, y, q) \in BR(M, \theta)$. Then
i) $(m, x, n)\, \mathcal{R}\, (p, y, q) \Leftrightarrow m = p$
ii) $(m, x, n) \mathcal{L}\, (p, y, q) \Leftrightarrow n = q$

Asibong-Ibe [1] considered the *-bisimple ample ω-semigroup and proved that they are isomorphic to certain generalized Bruck-Reilly extension $BR^*(M, \theta)$ of a cancellative monoid M where θ is a morphism. Below are some of his results.

Proposition 3.4 [1]. Let M be a cancellative monoid with identity e and $\theta : M \to M$ be a morphism. Let $(0, e, 0)$ be the identity of $BR^*(M, \theta)$. Then for $(m, x, n), (p, y, q) \in BR^*(M, \theta)$
i) $(m, x, n)\, \mathcal{R}^*(p, y, q) \Leftrightarrow m = p$
ii) $(m, x, n)\, \mathcal{L}^*(p, y, q) \Leftrightarrow n = q$

Proposition 3.5 [1]. $BR^*(M, \theta)$ is left ample.

Yu Shang and Limin Wang [7] considered a similar construction of the Bruck-Reilly extension of a monoid. They used this construction to give a structure theorem for *-bisimple ample I-semigroup. Below are some of their results.

Construction 3.6. [7]. Let M be a monoid with identity e and $\theta : M \to M$ be a morphism. The set $S = GBR^*(M, \theta) = I \times M \times I$ (where I denotes a non-empty set) with multiplication defined by the rule

$$(m, x, n)(P, y, q) = \begin{cases} (m, x.f_{n-p,p}^{-1}.y\theta^{n-p}.f_{n-p,q}, q - p + n) & \text{if } n \geq p \\ (m - n + p, f_{p-n,m}^{-1}.x\theta^{p-n}.f_{p-n,n}.y, q) & \text{if } n \leq p \end{cases}$$

(where θ^0 is the identity map on $M, f_{0,n} = e$ is the identity of M) forms a semigroup. This semigroup is called the generalized Bruck-Reilly *-extension of M determined by θ.

Remark 3.7. The idempotents of $GBR^*(M, \theta)$ are of the form (m, e, m), where $m \in \mathbb{N}^0$.

Lemma 3.8 [7]. Let $(m, x, n), (p, y, q) \in GBR^*(M, \theta)$. Then
i) $(m, x, n)\, \mathcal{L}^*(p, y, q) \Leftrightarrow n = q$
ii) $(m, x, n)\, \mathcal{R}^*(p, y, q) \Leftrightarrow m = p$

4. $\tilde{\mathcal{J}}_E$ - SIMPLE LEFT RESTRICTION ω-SEMIGROUP

In this section, we show that $BR(M, \theta)$ is a $\tilde{\mathcal{J}}_E$- simple left restriction ω-semigroup. But first, we need the definition of a strong semilattice of monoids, which is taken from [6].

Definition 4.1. Let M be a semigroup which is the disjoint union of monoids M_α, where the indices α form a semilattice Y. Suppose that for all $\alpha, \beta, \gamma,\ M_\alpha M_\beta \subseteq M_{\alpha\beta}$. Then M is called a semilattice Y of monoids M_α where $\alpha \in Y$. Furthermore, consider $\alpha, \beta \in Y$ where $\alpha \geq \beta$. Let $\varphi_{\alpha,\beta} : M_\alpha \to M_\beta$ be a monoid morphism such that :

i) $\varphi_{\alpha,\alpha} = \mathrm{id}_{M_\alpha}$ for all $\alpha \in Y$

ii) for $\alpha,\beta,\gamma \in Y$, where $\alpha \geq \beta \geq \gamma$, $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

Then $\varphi_{\alpha,\beta}$ is called a connecting morphism. Furthermore, if for all $x, y \in M$ where $x \in M_\alpha$ and $y \in M_\beta$ we have that

$$x \,.\, y = (x\varphi_{\alpha,\alpha\beta})(y\varphi_{\beta,\alpha\beta})$$

Then $M = [\,Y,\, M_\alpha;\, \varphi_{\alpha,\beta}]$ is called a strong semilattice Y of monoids M_α with connecting morphisms $\varphi_{\alpha,\beta}$.

It can be easily shown that $(M,.)$ is a semigroup with identity e_0 and the multiplication on M extends the multiplication in each M_α.

Proposition 4.2. Let $M = \bigcup_{\alpha=0}^{d-1} M_\alpha$ be a strong semilattice of the monoids M_α where $d \in \mathbb{N}^0$, the indices α form a chain $0 > 1 > \cdots > d-1$ and the connecting morphisms are all monoid morphisms. Let $\theta : M \to M_0$ be a monoid morphism and $E = \{\,(m, e_\alpha, m) : m \in \mathbb{N}^0, 0 \leq \alpha \leq d-1\}$ where e_α is the identity of M_α. Then for any $(m,x,n),(p,y,q) \in BR(M,\theta)$ we have

i) $(m,x,n)\,\widetilde{\mathcal{R}}_E(p,y,q) \Leftrightarrow m = p$, and $x, y \in M_\alpha$.
ii) $(m,x,n)\widetilde{\mathcal{L}}_E(p,y,q) \Leftrightarrow n = q$, and $x, y \in M_\alpha$.
iii) $(m,x,n)\,\widetilde{\mathcal{D}}_E(p,y,q) \Leftrightarrow x, y \in M_\alpha$, so $BR(M,\theta)$ has d $\widetilde{\mathcal{D}}_E$ - classes.
iv) $(m,x,n)\widetilde{\mathcal{J}}_E(p,y,q)$. That is, $\widetilde{\mathcal{J}}_E$ is the universal relation.

Proof. i)
$\Longrightarrow$ Let $(m,x,n)\,\widetilde{\mathcal{R}}_E(p,y,q)$, where $x \in M_\alpha$ and $y \in M_\beta$ for some α and β. Then for $(m,e_\alpha,m) \in E$
$(m,e_\alpha,m)(m,x,n) = (m-m+t_1, e_\alpha\theta^{t_1-m}x\theta^{t_1-m}, n-m+t_1) = (m,x,n)$
where $t_1 = \max\,(m,m) = m$
$(m,e_\alpha,m)(p,y,q) = (m-m+t_2, e_\alpha\theta^{t_2-m}y\theta^{t_2-p}, q-p+t_2) = (p,y,q)$
where $t_2 = \max\,(m,p)$
If $m \leq p$, this gives
$$(t_2, e_\alpha\theta^{t_2-m}y\theta^{t_2-p}, q-p+t_2) = (p,y,q)$$
Comparing the first coordinates gives $t_2 = p$
Similarly if $p \leq m$, this gives
$$(t_2, e_\alpha y\theta^{t_2-p}, q-p+t_2) = (p,y,q)$$
Comparing the first coordinates gives $t_2 = m = p$
So we have that $m = p \Longrightarrow e_\alpha\theta^{t_2-m} = e_\alpha$.
We know that $e_\alpha \in M_\alpha$, $y \in M_\beta$, so $e_\alpha y \in M_{\max(\alpha,\beta)} \Longrightarrow \max\,(\alpha,\beta) = \beta$, that is $e_\beta x = x \Longrightarrow \beta \leq \alpha$. Thus $m = p$ and $\alpha = \beta$.
$\Longleftarrow$ Let $m = p$, $x, y \in M_\alpha$ and $(l, e_\beta, l) \in E$ be such that
$$(l,e_\beta,l)(m,x,n) = (l-l+t_3, e_\beta\theta^{t_3-l}x\theta^{t_3-m}, n-m+t_3) = (m,x,n)$$
where $t_3 = \max\,(l,m)$
Then necessarily $l \leq m$ and $\beta \leq \alpha$.
$$(l,e_\beta,l)(m,y,q) = (m, e_\beta\theta^{m-l}y, q) = (m,y,q)$$
Similarly, it is easy to see that for $(k,e_\beta,k) \in E$, we have
$$(k,e_\beta,k)(p,y,q) = (p,y,q),$$
$$(k,e_\beta,k)(p,x,n) = (p,x,n).$$
Thus $(m,x,n)\,\widetilde{\mathcal{R}}_E(p,y,q)$.
ii) The proof is similar to i).
iii)
$\Longrightarrow$ Let $(m,x,n)\,\widetilde{\mathcal{D}}_E(p,y,q)$. Then there exists an element $(m,x,q) \in BR(M,\theta)$ such that
$$(m,x,n)\,\widetilde{\mathcal{R}}_E(m,z,q)\,\widetilde{\mathcal{L}}_E(p,y,q)$$
Clearly, it follows that $x, y, z \in M_\alpha$ for some α.
$\Longleftarrow$ Let $x, y \in M_\alpha$, then clearly we have
$$(m,x,n)\,\widetilde{\mathcal{R}}_E(m,x,q)\,\widetilde{\mathcal{L}}_E(p,y,q)$$
Thus $(m,x,n)\,\widetilde{\mathcal{D}}_E(p,y,q)$.
iv) Let $(m,x,n),(p,y,q) \in BR(M,\theta)$ where $x \in M_\alpha$ and $y \in M_\beta$. Then we have
$$(p,e_\beta,m+1)(m,x,n) = (p-(m+1)+t, e_\beta\theta^{t-m-1}x\theta^{t-m}, n-m+t)$$
$$= (p, e_\beta(x\theta), n+1)$$
where $t = \max(m+1,m) = m+1$. Obviously $e_\beta(x\theta) \in M_\beta$. Then
$$(p, e_\beta(x\theta), n+1)\,\widetilde{\mathcal{D}}_E(p,y,q)$$
In the same way $(m,e_\alpha,p+1)(p,y,q)\,\widetilde{\mathcal{D}}_E(m,x,n)$.
Thus $(m,x,n)\,\widetilde{\mathcal{J}}_E\,(p,y,q)$. It now follows from Lemma 2.5[4] that $BR(M,\theta)$ is $\widetilde{\mathcal{J}}_E$- simple.

Proposition 4.3. $BR(M,\theta)$ is left restriction
Proof. We have to check that the conditions of Definition 2.3 hold.
First we show that the elements of E commute. So for $m,n \in \mathbb{N}^0$, we have
$(m,e_\alpha,m)(n,e_\beta,n) = (m-m+t, e_\alpha\theta^{t-m}e_\beta\theta^{t-n}, n-n+t)$

$= (t, e_\alpha \theta^{t-m} e_\beta \theta^{t-n}, t) = (t, e_\alpha \theta^{t-n} e_\beta \theta^{t-m}, t)$
$= (n, e_\beta, n)(m, e_\alpha, m)$
where $t = \max(m, n)$.
To show that $(m, x, n)\, \widetilde{\mathcal{R}}_E\, (m, e_\alpha, m)$, we have
$(m, e_\alpha, m)(m, x, n) = (m - m + t_1, e_\alpha \theta^{t_1 - m} x \theta^{t_1 - m}, n - m + t_1)$
$= (t_1, e_\alpha \theta^0 x \theta^0, n) = (m, x, n)$
where $t_1 = \max(m, m) = m$
For $(p, e_\beta, p) \in E$,
$(p, e_\beta, p)(m, x, n) = (m, x, n) \Rightarrow (p - p + t_2, e_\beta \theta^{t_2 - p} x \theta^{t_2 - m}, n - m + t_2)$
$= (t_2, x\theta^{t_2 - m}, n - m + t_2) = (m, x, n)$, $t_2 = \max(p, m)$
$\Rightarrow t_2 = m$,
$\Rightarrow (p, e_\beta, p)(m, e_\alpha, m) = (m, e_\alpha, m)$
So $(m, x, n)\, \widetilde{\mathcal{R}}_E\, (m, e, m)$ and we let $(m, x, n)^\dagger = (m, e_\alpha, m)$.
To show that $\widetilde{\mathcal{R}}_E$ is a left congruence, let $(m, x, n), (p, y, q) \in BR(M, \theta)$
$(m, x, n)\, \widetilde{\mathcal{R}}_E\, (p, y, q) \Leftrightarrow (m, x, n)^\dagger = (p, y, q)^\dagger$
$\Leftrightarrow (m, e_\alpha, m) = (p, e_\beta, p)$
$\Leftrightarrow m = p$
So $(m, x, n)\, \widetilde{\mathcal{R}}_E\, (p, y, q) \Rightarrow m = p$
$\Rightarrow \max(z, m) = \max(z, p)$, for $z \in \mathbb{N}^0$
$\Rightarrow k - z + \max(z, m) = k - z + \max(z, p)$, for $k, z \in \mathbb{N}^0$
$\Rightarrow \big((k, c, z)(m, x, n)\big)^\dagger = \big((k, c, z)(p, y, q)\big)^\dagger$
$\Rightarrow (k, c, z)(m, x, n)\, \widetilde{\mathcal{R}}_E (k, c, z)(p, y, q)$
for any $(k, c, z) \in BR(M, \theta)$. Thus $\widetilde{\mathcal{R}}_E$ is a left congruence.
By Proposition 3.5[1], the left ample condition hold and so $BR(M, \theta)$ is left restriction.

Let $E = \{f_\alpha : \alpha \in \mathbb{N}^0\}$ be the distinguished semilattice of idempotents, where $f_\alpha \leq f_\beta \Leftrightarrow \alpha \geq \beta$ for all $\alpha, \beta \in \mathbb{N}^0$. Then E is called C_ω, that is, C_ω is a descending chain

$$f_0 > f_1 > f_2 > \cdots$$

If S is a left (right) restriction with distinguished semilattice of idempotents E, then S is said to be an ω-semigroup if E is isomorphic to C_ω .

Proposition 4.4. $BR(M, \theta)$ is an ω-semigroup
Proof. Let $(m, e_\alpha, m), (n, e_\beta, n) \in E$ where $m > n$. Then

$$(m, e_\alpha, m)(n, e_\beta, n) = (m, e_\alpha(e_\beta \theta^{m-n}), m) = (m, e_\alpha, m)$$

since $(e_\beta \theta^{m-n})$ is the identity of M, so we have $(m, e_\alpha, m) < (n, e_\beta, n)$. Also if $m = n$ and $\alpha \geq \beta$, then we have

$$(m, e_\alpha, m)(m, e_\beta, m) = (m, e_\alpha e_\beta, m) = (m, e_\alpha, m)$$

So that $(m, e_\alpha, m) \geq (m, e_\beta, m) \Leftrightarrow m < n$, or if $m = n$ and $\alpha \leq \beta$. So E is the chain

$$(0, e_0, 0) > (0, e_1, 0) > \cdots > (0, e_{d-1}, 0)$$
$$> (1, e_0, 1) > (1, e_1, 1) > \cdots > (1, e_{d-1}, 1)$$
$$> (2, e_0, 2) > (2, e_1, 2) > \cdots > (2, e_{d-1}, 2)$$
$$> \cdots$$

Thus $BR(M, \theta)$ is a $\widetilde{\mathcal{J}}_E$ -simple left restriction ω-semigroup.

Hence, we get the following conclusion:

Theorem 4.5. Let $M = \bigcup_{\alpha=0}^{d-1} M_\alpha$ be a strong semilattice of the monoids M_α where $d \in \mathbb{N}^0$, the indices α form a chain $0 > 1 > \cdots > d-1$ and the connecting morphisms are all monoid morphisms. Let $\theta : M \to M_0$ be a monoid morphism and $E = \{ (m, e_\alpha, m) : m \in \mathbb{N}^0, 0 \leq \alpha \leq d-1\}$ where e_α is the identity of M_α. Then $BR(M, \theta)$ is a $\widetilde{\mathcal{J}}_E$ -simple left restriction ω-semigroup.

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