On Quasi-Left Primary and Quasi-Primary Γ-ideals in Γ-AG-Groupoids

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ABSTRACT— ***The purpose of this paper is to introduce the notion of a quasi-primary ideals in Γ-AG-groupoids, we study quasi-primary and quasi-left primary ideals in Γ-AG-groupoids. Some characterizations of quasi-primary and quasi-left primary ideals are obtained. Moreover, we investigate the relationships between quasi-primary and quasi-left primary ideals in Γ-AG-groupoids. Finally, we obtain*** *the* ***necessary and sufficient conditions of a quasi-primary ideal to be a quasi-left primary ideal in Γ-AG-groupoids.***

Keywords— Γ-AG-groupoid, Γ-LA-semigroup, Γ-ideal, quasi-primary ideal, quasi-left primary ideal.

1. INTRODUCTION

Abel-Grassmann's groupoid (AG-groupoid) is a generalization of semigroup theory with wide range of usages in theory of flocks [6]. The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A groupoid S is called an AG-groupoid if it satisfies the left invertive law:

$$(ab)c = (cb)a$$

for all $a,\ b,\ c \in S$. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid [5]. This structure is closely related to a commutative semigroup. Becauseof containing a right identity, an AG-groupoid becomes a commutative monoid [5]. A left identity in an AG-groupoid is unique [5]. It lies between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [6]. Ideals in AG-groupoids have been discussed in [4]. In 1981, the notion of Γ-semigroups was introduced by M. K. Sen. A groupoid S is called a Γ-AG-groupoid if it satisfies the left invertive law:

$$(a\gamma b)\delta c = (c\gamma b)\delta a$$

for all $a,b,c \in S$ and $\gamma,\delta \in \Gamma$ [3]. This structure is also known as a left almost semigroup (LA-semigroup). In this paper, we are going to investigate some interesting properties of recently discovered classes, namely Γ-AG-groupoid S always satisfies the Γ-medial law:

$$(a\gamma b)\beta(c\delta d) = (a\gamma c)\beta(b\delta d)$$

for all $a,b,c,d \in S$ and $\gamma,\beta,\delta \in \Gamma$ [3], while a Γ-AG-groupoid S with left identity e always satisfies Γ-paramedial law:

$$(a\gamma b)\beta(c\delta d) = (d\gamma c)\beta(b\delta a)$$

for all $a,b,c,d\in S$ and $\gamma,\beta,\delta\in\Gamma$ [3]. Recently T. Shah and I. Rehman have discussed Γ-Ideals and Γ-Bi-Ideals in Γ-AG-Groupoids.

In this paper we characterize the Γ-AG-groupoid. We investigate the relationships between quasi-primary and quasi-left primary ideals in Γ-AG-groupoids.

2. BASIC PROPERTIES

In this section we refer to [10, 11, 12, 13] for some elementary aspects and quote few definitions, and essential examples to step up this study. For more details we refer to the papers in the references.

Example 2.1. [10, 11] (1). Let S be an arbitrary AG-groupoid and Γ any non-empty set. Define a mapping $S\times\Gamma\times S\rightarrow S$; by $a\gamma b=ab$ for all $a,b\in S$ and $\gamma\in\Gamma$. It is easy to see that S is a Γ-AG-groupoid.
(2). Let $\Gamma=\{1,2,3\}$. Define a mapping $\mathbb{Z}\times\Gamma\times\mathbb{Z}\rightarrow\mathbb{Z}$ by $a\gamma b=b-\gamma-a$ for all $a,b\in\mathbb{Z}$ and $\gamma\in\Gamma$ where "$-$" is a usual subtraction of integers. Then $\mathbb{Z}$ is a Γ-AG-groupoid.

Lemma 2.2. [10, 11] Every Γ-AG-groupoid is Γ-medial.

Lemma 2.3. [10, 11] Let S be a Γ-AG-groupoid with a left identity, then $a\gamma(b\alpha c)=b\gamma(a\alpha c)$ for all $a,b,c\in S$ and $\gamma,\alpha\in\Gamma$.

Definition 2.4. [10, 11] Let S be a Γ-AG-groupoid. A nonempty subset A of S is called a sub Γ-AG-groupoid of S if $A\Gamma A\subseteq A$.

Definition 2.5. [10, 11] A sub Γ-AG-groupoid A of S is called a left (right) Γ-ideal of S if $S\Gamma A\subseteq A$ ($A\Gamma S\subseteq A$) and is called an Γ-ideal if it is left as well as right Γ-ideal.

Lemma 2.6. [10, 11] If a Γ-AG-groupoid S has a left identity, then every right Γ-ideal is a left Γ-ideal.

Lemma 2.7. [10, 11] If A is a left Γ-ideal of a Γ-AG-groupoid S with left identity, and if for any $a\in S$, there exists $\gamma\in\Gamma$, then $a\gamma A$ is a left Γ-ideal of S.

Lemma 2.8. [10, 11] If A is a proper right (left) Γ-ideal of a Γ-AG-groupoid S with left identity e, then $e\notin A$.

Lemma 2.9. [13] If S is a Γ-AG-groupoid with left identity e, then $a\gamma b=a\beta b$ for all $a,b\in S$ and $\gamma,\beta\in\Gamma$.

Lemma 2.10. Let S be a Γ-AG-groupoid with left identity, and let B be a left Γ-ideal of S. Then $A\Gamma B=\{a\gamma b : a\in A, b\in B, \gamma\in\Gamma\}$ is a left Γ-ideal in S, where $\varnothing\neq A\subseteq S$.

Lemma 2.11. Let S be a Γ-AG-groupoid with left identity and let $a \in S$. Then

$$a^2\gamma S = \left\{a^2\gamma s = (a\beta a)\gamma s : s \in S\right\}$$

is a Γ-ideal in S, where $\gamma, \beta \in \Gamma$.

Lemma 2.12. Let S be a Γ-AG-groupoid with left identity, and let A be a left Γ-ideal of S. Then $(A:\gamma:r)$ is a left Γ-ideal in S, where $(A:\gamma:r) = \{a \in S : r\gamma a \in A\}$.

Remark. Let S be a Γ-AG-groupoid and let A be a left Γ-ideal of S .It is easy to verify that $A \subseteq (A:\gamma:r)$.

Lemma 2.13. Let S be a Γ-AG-groupoid with left identity, and let A, B be left Γ-ideals of S. Then $(A:\Gamma:B)$ is a left Γ-ideal in S, where $(A:\Gamma:B) = \{r \in S : B\Gamma r \subseteq A\}$.

Remark. Let S be a Γ-AG-groupoid and let A, B, C be left Γ-ideals of S .It is easy to verify that $(A:\Gamma:C) \subseteq (A:\Gamma:B)$, where $B \subseteq C$.

3. QUASI-LEFT PRIMARY AND LEFT PRIMARY Γ-IDEALS

We start with the following theorem that gives a relation between Γ-primary and quasi Γ-primary ideal in Γ-AG-groupoid. Our starting points are the following definitions:

Definition 3.1. A Γ-ideal P is called left quasi-primary if $A\Gamma B \subseteq P$ implies that

$$\left(\left((A\Gamma A)\Gamma A\right)\Gamma \ldots\right)\Gamma A = A^n \subseteq P \text{ or } \left(\left((B\Gamma B)\Gamma B\right)\Gamma \ldots\right)\Gamma B = B^n \subseteq P$$

for some positive integer n, where A and B are two Γ-ideals of S.

Definition 3.2. A left Γ-ideal P is called quasi-left primary if $A\Gamma B \subseteq P$ implies that

$$\left(\left((A\Gamma A)\Gamma A\right)\Gamma \ldots\right)\Gamma A = A^n \subseteq P \text{ or } \left(\left((B\Gamma B)\Gamma B\right)\Gamma \ldots\right)\Gamma B = B^n \subseteq P$$

for some positive integer n, where A and B are two left Γ-ideals of S.

Remark. It is easy to see that every quasi-left primary ideal is quasi-primary.

Lemma 3.3. If S is a Γ-AG-groupoid with left identity, then a left Γ-ideal P of S is quasi-left primary if and only if $a\gamma(S\beta b)\subseteq P$ implies that

$$(((a\delta a)\delta a)\delta\ldots)\delta a=a^{n}\in P \text{ or } (((b\delta b)\delta b)\delta\ldots)\delta b=b^{n}\in P$$

for some positive integer n, where $\gamma,\beta,\delta\in\Gamma$ and $a,b\in S.$

Proof. Let P be a quasi-left primary left ideal of a Γ-AG-groupoid S with left identity. Now suppose that $a\gamma(S\beta b)\subseteq P.$ Then by Definition of left Γ-ideal, we get $S\Gamma(a\gamma(S\beta b))\subseteq S\Gamma P\subseteq P$ that is,

$$\begin{aligned}
S\Gamma(a\gamma(S\beta b)) &= (S\delta S)\Gamma(a\gamma(S\beta b))\\
&= (S\delta a)\Gamma(S\gamma(S\beta b))\\
&= (S\delta a)\Gamma((S\Gamma S)\gamma(S\beta b))\\
&= (S\delta a)\Gamma((b\beta S)\gamma(S\Gamma S))\\
&= (S\delta a)\Gamma((b\beta S)\gamma(S))\\
&= (S\delta a)\Gamma((S\beta S)\gamma b)\\
&= (S\delta a)\Gamma(S\gamma b)
\end{aligned}$$

for all $\delta\in\Gamma$. Since $S\Gamma(a\gamma(S\beta b))\subseteq P$ and $S\Gamma(a\gamma(S\beta b))=(S\delta a)\Gamma(S\gamma b),$ we have $(S\delta a)\Gamma(S\gamma b)\subseteq P$ so that $a^{n}=(e\delta a)^{n}\in(S\delta a)^{n}\subseteq P$ or $b^{n}=(eb)^{n}\in(S\gamma b)^{n}\subseteq P,$ for some positive integer n. Conversely, assume that if $a\gamma(S\beta b)\subseteq P$ implies that $a^{n}\in P$ or $b^{n}\in P$ for some positive integer n, where $\gamma,\beta\in\Gamma$ and $a,b\in S.$ Suppose that $A\Gamma B\subseteq P$, where A and B are left Γ-ideals of S such that $A\not\subset P$. Then there exists $x\in A$ such that $x^{n}\notin P,$ for all positive integer n. Now

$$x\gamma(S\beta y)\subseteq A\Gamma(S\Gamma B)\subseteq A\Gamma B\subseteq P,$$

for all $y\in B$. So by hypothesis, $y^{n}\in P$ for all $y\in B$ implies that $B^{n}\subseteq P$. Hence P is quasi-left primary ideal in $S.$

Lemma .3.4. If S is a Γ-AG-groupoid with left identity, then a left Γ-ideal P of S is quasi-left primary if and only if $(S\gamma a)\delta(S\beta b)\subseteq P$ implies that $a^{n}\in P$ or $b^{n}\in P$ for some positive integer n, where $\gamma,\beta,\delta\in\Gamma$ and $a,b\in S.$

Proof. Let P be a quasi-left primary ideal of a Γ-AG-groupoid S with left identity. Now suppose that $(S\gamma a)\delta(S\beta b)\subseteq P$. Then by Definition of left ideal, we get

$$(S\gamma a)\delta(S\beta b) = (S\gamma S)\delta(a\beta b)$$

$$= S\delta(a\beta b)$$

$$= a\delta(S\beta b)$$

that is $a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P.$ By Lemma 3.3, we have $a^n \in P$ or $b^n \in P$ for some positive integer n. Conversely, assume that if $(S\gamma a)\delta(S\beta b) \subseteq P$, then $a^n \in P$ or $b^n \in P$ for some positive integer n, where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S.$ Let $a\delta(S\beta b) \subseteq P.$ Now consider

$$a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P.$$

By using given assumption, if $a\delta(S\beta b) \subseteq P$, then $a^n \in P$ or $b^n \in P$ for some positive integer n. Then by Lemma 3.3, we have P is a quasi-left primary ideal in $S.$

Theorem 3.5. If S is a Γ-AG-groupoid with left identity, then a left Γ-ideal P of S is quasi-left primary if and only if $a\gamma b \in P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer n, where $\gamma \in \Gamma$ and $a, b \in S.$

Proof. Let P be a left Γ-ideal of a Γ-AG-groupoid S with left identity. Now suppose that $a\gamma b \in P.$ Then by Definition of left ideal, we get

$$\begin{aligned}(S\alpha a)\beta(S\gamma b) &= (S\alpha S)\beta(a\gamma b)\\ &= S\beta(a\gamma b)\\ &\subseteq S\Gamma P\\ &\subseteq P.\end{aligned}$$

By Lemma 3.4, we have $a^n \in P$ or $b^n \in P$ for some positive integer n. Conversely, the proof is easy.

Theorem 3.6. Let S be a Γ-AG-groupoid, and let A be a quasi-left primary ideal of $S.$ Then $(A:\gamma:r)$ is a quasi-left primary ideal in S, where $\gamma \in \Gamma$ and $r \in S.$

Proof. Assume that A is a quasi-left primary ideal of $S.$ By Lemma 2.12, we have $(A:\gamma:r)$ is a left ideal in $S.$ Let $a\beta b \in (A:\gamma:r).$ Suppose that $b^n \notin (A:\gamma:r)$, for all positive integer n. Since $a\beta b \in (A:\gamma:r)$, we have $r\gamma(a\beta b) \in A$ so that $a\gamma(r\beta b) \in A$. By Theorem 3.5, we have $a^n \in A \subseteq (A:\gamma:r)$ or $(r\beta b)^n \in A$, for some positive integer n. Therefore $a^n \in (A:\gamma:r)$ and hence $(A:\gamma:r)$ is a quasi-left primary ideal in $S.$

Theorem 3.7. Let S be a Γ-AG-groupoid with left identity e and let P be a quasi-primary ideal of S. If $(S\gamma a^2)\alpha(S\beta b^2)\subseteq P$, then $a^n\in P$ or $b^n\in P$, for some positive integer n, where $\gamma\in\Gamma$ and $a,b\in S$.

Proof. Let P be a quasi-primary ideal of a Γ-AG-groupoid S with left identity. Suppose that $b^n\notin P$, for all positive integer n. Now assume that $(S\gamma a^2)\alpha(S\beta b^2)\subseteq P$. Then by Definition of left Γ-ideal, we get

$$\begin{aligned}
(S\gamma a^2)\alpha(S\beta b^2) &= ((S\beta b^2)\gamma a^2)\alpha S\\
&= ((a^2\beta b^2)\gamma S)\alpha S\\
&= (S\gamma S)\alpha(a^2\beta b^2)\\
&= a^2\alpha((S\gamma S)\beta b^2)\\
&= a^2\alpha((b^2\gamma S)\beta S)\\
&= (b^2\gamma S)\alpha(a^2\beta S)
\end{aligned}$$

that is $(b^2\gamma S)\alpha(a^2\beta S)\subseteq P$. By Lemma 2.11, we have $a^2\beta S$ and $b^2\gamma S$ are Γ-ideals in S so that

$$\begin{aligned}
a^2 &= a\lambda a\\
&= (e\chi a)\lambda a\\
&= (a\chi a)\lambda e\\
&= (a\lambda a)\beta e\\
&= a^2\beta e\in a^2\beta S\subseteq P
\end{aligned}$$

or

$$\begin{aligned}
b^2 &= b\lambda b\\
&= (e\chi b)\lambda b\\
&= (b\chi b)\lambda e\\
&= (b\lambda b)\gamma e\\
&= b^2\gamma e\in b^2\gamma S\subseteq P
\end{aligned}$$

for all $\chi\in\Gamma$. Therefore $a^n\in P$, for some positive integer n.

Corollary 3.8. Let S be a Γ-AG-groupoid with left identity, and let P be a quasi-primary ideal of S. If $b^2\gamma a^2\in P$, then $a^n\in P$ or $b^n\in P$, for some positive integer n.

Proof. Let P be a quasi-primary ideal of an AG-groupoid S with left identity. Suppose that $b^n\notin P$, for all positive integer n. Now assume that $b^2\gamma a^2\in P$. Then by Definition of left Γ-ideal, we get

$$
\begin{aligned}
\left(a^2\beta S\right)\alpha\left(b^2\gamma S\right) &= b^2\alpha\left(\left(a^2\beta S\right)\gamma S\right)\\
&= b^2\alpha\left(\left(S\beta S\right)\gamma a^2\right)\\
&= \left(S\beta S\right)\alpha\left(b^2\gamma a^2\right)\\
&= S\alpha\left(b^2\gamma a^2\right)\\
&\subseteq S\Gamma P\\
&\subseteq P
\end{aligned}
$$

that is $\left(a^2\beta S\right)\alpha\left(b^2\gamma S\right)\subseteq P$. It is easy to see that $a^n \in P,$ for some positive integer n.

Definition 3.9. A Γ-AG-groupoid S is called Γ-AG-3-band if its every element satisfies

$$a\alpha\left(a\beta a\right)=\left(a\alpha a\right)\beta a = a.$$

Proposition 3.10. [13] Every left identity in a Γ-AG-3-band is a right identity.

Lemma 3.11. [13] If a Γ-AG-3-band S has a left identity, then every left Γ-ideal is a Γ-ideal.

Theorem 3.12. Let S be a Γ-AG-3-band with left identity. Then P is a quasi-left primary ideal in S if and only if S is a quasi-primary ideal in S.

Proof. The proof is straightforward. □

ACKNOWLEDGEMENT

The authors are very grateful to the anonymous referee for stimulating comments and improving presentation of the paper.

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