On Quasi-Left Primary and Quasi-Primary $\Gamma$-ideals in $\Gamma$-AG-Groupoids

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ABSTRACT— The purpose of this paper is to introduce the notion of a quasi-primary ideals in $\Gamma$-AG-groupoids, we study quasi-primary and quasi-left primary ideals in $\Gamma$-AG-groupoids. Some characterizations of quasi-primary and quasi-left primary ideals are obtained. Moreover, we investigate the relationships between quasi-primary and quasi-left primary ideals in $\Gamma$-AG-groupoids. Finally, we obtain the necessary and sufficient conditions of a quasi-primary ideal to be a quasi-left primary ideal in $\Gamma$-AG-groupoids.

Keywords— $\Gamma$-AG-groupoid, $\Gamma$-LA-semigroup, $\Gamma$-ideal, quasi-primary ideal, quasi-left primary ideal.

1. INTRODUCTION

Abel-Grassmann’s groupoid (AG-groupoid) is a generalization of semigroup theory with wide range of usages in theory of flocks [6]. The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A groupoid $S$ is called an AG-groupoid if it satisfies the left invertive law:

$$(ab)c = (cb)a$$

for all $a, b, c \in S$. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid [5]. This structure is closely related to a commutative semigroup. Because of containing a right identity, an AG-groupoid becomes a commutative monoid [5]. A left identity in an AG-groupoid is unique [5]. It lies between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [6]. Ideals in AG-groupoids have been discussed in [4]. In 1981, the notion of $\Gamma$-semigroups was introduced by M. K. Sen. A groupoid $S$ is called a $\Gamma$-AG-groupoid if it satisfies the left invertive law:

$$(\gamma \gamma \beta \delta) = (\gamma \beta \delta)$$

for all $a, b, c, d \in S$ and $\gamma, \delta \in \Gamma$ [3]. This structure is also known as a left almost semigroup (LA-semigroup). In this paper, we are going to investigate some interesting properties of recently discovered classes, namely $\Gamma$-AG-groupoid $S$ always satisfies the $\Gamma$-medial law:

$$(\gamma \beta \delta) = (\gamma \beta \delta)$$

for all $a, b, c, d \in S$ and $\gamma, \beta, \delta \in \Gamma$ [3], while a $\Gamma$-AG-groupoid $S$ with left identity $e$ always satisfies $\Gamma$-paramedial law:

$$(\gamma \beta \delta) = (\gamma \beta \delta)$$
for all \( a,b,c,d \in S \) and \( \gamma, \beta, \delta \in \Gamma \) \([3]\). Recently T. Shah and I. Rehman have discussed \( \Gamma \)-Ideals and \( \Gamma \)-Bi-Ideals in \( \Gamma \)-AG-Groupoids.

In this paper we characterize the \( \Gamma \)-AG-groupoid. We investigate the relationships between quasi-primary and quasi-left primary ideals in \( \Gamma \)-AG-groupoids.

2. BASIC PROPERTIES

In this section we refer to \([10, 11, 12, 13]\) for some elementary aspects and quote few definitions, and essential examples to step up this study. For more details we refer to the papers in the references.

**Example 2.1.** \([10, 11]\) (1). Let \( S \) be an arbitrary AG-groupoid and \( \Gamma \) any non-empty set. Define a mapping \( S \times \Gamma \times S \rightarrow S \) by \( a \gamma b = ab \) for all \( a,b \in S \) and \( \gamma \in \Gamma \). It is easy to see that \( S \) is a \( \Gamma \)-AG-groupoid.

(2). Let \( \Gamma = \{1,2,3\} \). Define a mapping \( \square \times \Gamma \times \square \rightarrow \square \) by \( a \gamma b = b - \gamma - a \) for all \( a,b \in \square \) and \( \gamma \in \Gamma \) where \( "-" \) is a usual subtraction of integers. Then \( \square \) is a \( \Gamma \)-AG-groupoid.

**Lemma 2.2.** \([10, 11]\) Every \( \Gamma \)-AG-groupoid is \( \Gamma \)-medial.

**Lemma 2.3.** \([10, 11]\) Let \( S \) be a \( \Gamma \)-AG-groupoid with a left identity, then \( a \gamma (b \alpha c) = b \gamma (a \alpha c) \) for all \( a,b,c \in S \) and \( \gamma, \alpha \in \Gamma \).

**Definition 2.4.** \([10, 11]\) Let \( S \) be a \( \Gamma \)-AG-groupoid. A nonempty subset \( A \) of \( S \) is called a sub \( \Gamma \)-AG-groupoid of \( S \) if \( A \Gamma A \subseteq A \).

**Definition 2.5.** \([10, 11]\) A sub \( \Gamma \)-AG-groupoid \( A \) of \( S \) is called a left (right) \( \Gamma \)-ideal of \( S \) if \( S \Gamma A \subseteq A \) (\( A \Gamma S \subseteq A \)) and is called an \( \Gamma \)-ideal if it is left as well as right \( \Gamma \)-ideal.

**Lemma 2.6.** \([10, 11]\) If a \( \Gamma \)-AG-groupoid \( S \) has a left identity, then every right \( \Gamma \)-ideal is a left \( \Gamma \)-ideal.

**Lemma 2.7.** \([10, 11]\) If \( A \) is a left \( \Gamma \)-ideal of a \( \Gamma \)-AG-groupoid \( S \) with left identity, and if for any \( a \in S \), there exists \( \gamma \in \Gamma \), then \( a \gamma A \) is a left \( \Gamma \)-ideal of \( S \).

**Lemma 2.8.** \([10, 11]\) If \( A \) is a proper right (left) \( \Gamma \)-ideal of a \( \Gamma \)-AG-groupoid \( S \) with left identity \( e \), then \( e \notin A \).

**Lemma 2.9.** \([13]\) If \( S \) is a \( \Gamma \)-AG-groupoid with left identity \( e \), then \( a \gamma b = a \beta b \) for all \( a,b \in S \) and \( \gamma, \beta \in \Gamma \).

**Lemma 2.10.** Let \( S \) be a \( \Gamma \)-AG-groupoid with left identity, and let \( B \) be a left \( \Gamma \)-ideal of \( S \). Then \( A \Gamma B = \{ a \gamma b : a \in A, b \in B, \gamma \in \Gamma \} \) is a left \( \Gamma \)-ideal in \( S \), where \( \emptyset \neq A \subseteq S \).
Lemma 2.11. Let $S$ be a $\Gamma$-AG-groupoid with left identity and let $a \in S$. Then
$$a^2 \gamma S = \{a^2 \gamma s = (a \beta a) \gamma s : s \in S\}$$
is a $\Gamma$-ideal in $S$, where $\gamma, \beta \in \Gamma$.

Lemma 2.12. Let $S$ be a $\Gamma$-AG-groupoid with left identity, and let $A$ be a left $\Gamma$-ideal of $S$. Then $(A : \gamma : r)$ is a left $\Gamma$-ideal in $S$, where $(A : \gamma : r) = \{a \in S : r \gamma a \in A\}$.

Remark. Let $S$ be a $\Gamma$-AG-groupoid and let $A$ be a left $\Gamma$-ideal of $S$. It is easy to verify that $A \subseteq (A : \gamma : r)$.

Lemma 2.13. Let $S$ be a $\Gamma$-AG-groupoid with left identity, and let $A, B$ be left $\Gamma$-ideals of $S$. Then $(A : \Gamma : B)$ is a left $\Gamma$-ideal in $S$, where $(A : \Gamma : B) = \{r \in S : B \gamma r \subseteq A\}$.

Remark. Let $S$ be a $\Gamma$-AG-groupoid and let $A, B, C$ be left $\Gamma$-ideals of $S$. It is easy to verify that $(A : \Gamma : C) \subseteq (A : \Gamma : B)$, where $B \subseteq C$.

3. QUASI-LEFT PRIMARY AND LEFT PRIMARY $\Gamma$-IDEALS

We start with the following theorem that gives a relation between $\Gamma$-primary and quasi $\Gamma$-primary ideal in $\Gamma$-AG-groupoid. Our starting points are the following definitions:

Definition 3.1. A $\Gamma$-ideal $P$ is called left quasi-primary if $A \Gamma B \subseteq P$ implies that
$$\left(\left(\left(\left(\left(\Gamma A\right) \Gamma A\right) \Gamma \ldots\right) \Gamma A\right) \Gamma \ldots\right) \Gamma A = A^n \subseteq P \text{ or } \left(\left(\left(\left(\left(\Gamma B\right) \Gamma B\right) \Gamma \ldots\right) \Gamma B\right) \Gamma \ldots\right) \Gamma B = B^n \subseteq P$$
for some positive integer $n$, where $A$ and $B$ are two $\Gamma$-ideals of $S$.

Definition 3.2. A left $\Gamma$-ideal $P$ is called quasi-left primary if $A \Gamma B \subseteq P$ implies that
$$\left(\left(\left(\left(\left(\Gamma A\right) \Gamma A\right) \Gamma \ldots\right) \Gamma A\right) \Gamma \ldots\right) \Gamma A = A^n \subseteq P \text{ or } \left(\left(\left(\left(\left(\Gamma B\right) \Gamma B\right) \Gamma \ldots\right) \Gamma B\right) \Gamma \ldots\right) \Gamma B = B^n \subseteq P$$
for some positive integer $n$, where $A$ and $B$ are two left $\Gamma$-ideals of $S$.

Remark. It is easy to see that every quasi-left primary ideal is quasi-primary.
Lemma 3.3. If $S$ is a $\Gamma$-AG-groupoid with left identity, then a left $\Gamma$-ideal $P$ of $S$ is quasi-left primary if and only if $a\gamma(S\beta b) \subseteq P$ implies that

$$((a\delta a)\delta a \ldots)\delta a = a^n \in P \text{ or } ((b\delta b)\delta b \ldots)\delta b = b^n \in P$$

for some positive integer $n$, where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$.

**Proof.** Let $P$ be a quasi-left primary left ideal of a $\Gamma$-AG-groupoid $S$ with left identity. Now suppose that $a\gamma(S\beta b) \subseteq P$. Then by Definition of left $\Gamma$-ideal, we get $\text{ST}(a\gamma(S\beta b)) \subseteq \text{ST}P \subseteq P$ that is,

$$\text{ST}(a\gamma(S\beta b)) = (S\delta S)\Gamma(a\gamma(S\beta b))$$

$$= (S\delta a)\Gamma(S\gamma(S\beta b))$$

$$= (S\delta a)\Gamma((S\Gamma S)\gamma(S\beta b))$$

$$= (S\delta a)\Gamma((b\beta S)\gamma(S\Gamma S))$$

$$= (S\delta a)\Gamma((b\beta S)\gamma(S))$$

$$= (S\delta a)\Gamma((S\beta S)\gamma b)$$

$$= (S\delta a)\Gamma(S\gamma b)$$

for all $\delta \in \Gamma$. Since $\text{ST}(a\gamma(S\beta b)) \subseteq P$ and $\text{ST}(a\gamma(S\beta b)) = (S\delta a)\Gamma(S\gamma b)$, we have $(S\delta a)\Gamma(S\gamma b) \subseteq P$ so that $a^n = (e\delta a)^n \in (S\delta a)^n \subseteq P$ or $b^n = (eb)^n \in (S\gamma b)^n \subseteq P$, for some positive integer $n$. Conversely, assume that if $a\gamma(S\beta b) \subseteq P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer $n$, where $\gamma, \beta \in \Gamma$ and $a, b \in S$. Suppose that $A\Gamma B \subseteq P$, where $A$ and $B$ are left $\Gamma$-ideals of $S$ such that $A \not\subseteq P$. Then there exists $x \in A$ such that $x^n \not\in P$, for all positive integer $n$. Now

$$x\gamma(S\beta y) \subseteq A\Gamma(S\Gamma B) \subseteq A\Gamma B \subseteq P,$$

for all $y \in B$. So by hypothesis, $y^n \in P$ for all $y \in B$ implies that $B^n \subseteq P$. Hence $P$ is quasi-left primary ideal in $S$.

Lemma 3.4. If $S$ is a $\Gamma$-AG-groupoid with left identity, then a left $\Gamma$-ideal $P$ of $S$ is quasi-left primary if and only if $(S\gamma a)\delta(S\beta b) \subseteq P$ implies that $a^n \in P$ or $b^n \in P$ for some positive integer $n$, where $\gamma, \beta, \delta \in \Gamma$ and $a, b \in S$.

**Proof.** Let $P$ be a quasi-left primary ideal of a $\Gamma$-AG-groupoid $S$ with left identity. Now suppose that $(S\gamma a)\delta(S\beta b) \subseteq P$. Then by Definition of left ideal, we get

$$(S\gamma a)\delta(S\beta b) = (S\gamma S)\delta(a\beta b)$$
that is \( a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P \). By Lemma 3.3, we have \( a^n \in P \) or \( b^n \in P \) for some positive integer \( n \).

Conversely, assume that if \( (S\gamma a)\delta(S\beta b) \subseteq P \), then \( a^n \in P \) or \( b^n \in P \) for some positive integer \( n \), where \( \gamma, \beta, \delta \in \Gamma \) and \( a, b \in S \). Let \( a\delta(S\beta b) \subseteq P \). Now consider

\[
a\delta(S\beta b) = (S\gamma a)\delta(S\beta b) \subseteq P.
\]

By using given assumption, if \( a\delta(S\beta b) \subseteq P \), then \( a^n \in P \) or \( b^n \in P \) for some positive integer \( n \). Then by Lemma 3.3, we have \( P \) is a quasi-left primary ideal in \( S \).

**Theorem 3.5.** If \( S \) is a \( \Gamma \)-AG-groupoid with left identity, then a left \( \Gamma \)-ideal \( P \) of \( S \) is quasi-left primary if and only if \( a\gamma b \in P \) implies that \( a^n \in P \) or \( b^n \in P \) for some positive integer \( n \), where \( \gamma \in \Gamma \) and \( a, b \in S \).

**Proof.** Let \( P \) be a left \( \Gamma \)-ideal of a \( \Gamma \)-AG-groupoid \( S \) with left identity. Now suppose that \( a\gamma b \in P \). Then by Definition of left ideal, we get

\[
(S\alpha a)\beta(S\gamma b) = (S\alpha S)\beta(a\gamma b)
\]

\[
= S\beta(a\gamma b)
\]

\[
\subseteq SP
\]

\[
\subseteq P.
\]

By Lemma 3.4, we have \( a^n \in P \) or \( b^n \in P \) for some positive integer \( n \). Conversely, the proof is easy.

**Theorem 3.6.** Let \( S \) be a \( \Gamma \)-AG-groupoid, and let \( A \) be a quasi-left primary ideal of \( S \). Then \( (A : \gamma : r) \) is a quasi-left primary ideal in \( S \), where \( \gamma \in \Gamma \) and \( r \in S \).

**Proof.** Assume that \( A \) is a quasi-left primary ideal of \( S \). By Lemma 2.12, we have \( (A : \gamma : r) \) is a left ideal in \( S \). Let \( a\beta b \in (A : \gamma : r) \). Suppose that \( b^n \notin (A : \gamma : r) \), for all positive integer \( n \). Since \( a\beta b \in (A : \gamma : r) \), we have \( r(\gamma(a\beta b)) \in A \) so that \( a\gamma(r\beta b) \in A \). By Theorem 3.5, we have \( a^n \in A \subseteq (A : \gamma : r) \) or \( (r\beta b)^n \in A \), for some positive integer \( n \). Therefore \( a^n \in (A : \gamma : r) \) and hence \( (A : \gamma : r) \) is a quasi-left primary ideal in \( S \).
Theorem 3.7. Let $S$ be a $\Gamma$-AG-groupoid with left identity $e$ and let $P$ be a quasi-primary ideal of $S$. If $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$, then $a^n \in P$ or $b^n \in P$, for some positive integer $n$, where $\gamma \in \Gamma$ and $a, b \in S$.

Proof. Let $P$ be a quasi-primary ideal of a $\Gamma$-AG-groupoid $S$ with left identity. Suppose that $b^n \notin P$, for all positive integer $n$. Now assume that $(S\gamma a^2)\alpha(S\beta b^2) \subseteq P$. Then by Definition of left $\Gamma$-ideal, we get

$$(S\gamma a^2)\alpha(S\beta b^2) = ((S\beta b^2)\gamma a^2)\alpha S$$

$$= ((a^2\beta b^2)\gamma S)\alpha S$$

$$= (S\gamma S)\alpha(a^2\beta b^2)$$

$$= a^2\alpha((S\gamma S)\beta b^2)$$

$$= a^2\alpha((b^2\gamma S)\beta S)$$

$$= (b^2\gamma S)\alpha(a^2\beta S)$$

that is $(b^2\gamma S)\alpha(a^2\beta S) \subseteq P$. By Lemma 2.11, we have $a^2\beta S$ and $b^2\gamma S$ are $\Gamma$-ideals in $S$ so that

$$a^2 = a\lambda a$$

$$= (e\chi a)\lambda a$$

$$= (a\lambda a)\lambda e$$

$$= (a\lambda a)\beta e$$

$$= a^2\beta e \in a^2\beta S \subseteq P$$

or

$$b^2 = b\lambda b$$

$$= (e\chi b)\lambda b$$

$$= (b\lambda b)\lambda e$$

$$= (b\lambda b)\gamma e$$

$$= b^2\gamma e \in b^2\gamma S \subseteq P$$

for all $\chi \in \Gamma$. Therefore $a^n \in P$, for some positive integer $n$.

Corollary 3.8. Let $S$ be a $\Gamma$-AG-groupoid with left identity, and let $P$ be a quasi-primary ideal of $S$. If $b^2\gamma a^2 \in P$, then $a^n \in P$ or $b^n \in P$, for some positive integer $n$.

Proof. Let $P$ be a quasi-primary ideal of an AG-groupoid $S$ with left identity. Suppose that $b^n \notin P$, for all positive integer $n$. Now assume that $b^2\gamma a^2 \in P$. Then by Definition of left $\Gamma$-ideal, we get
\[ (a^2 \beta S) \alpha (b^2 \gamma S) = b^2 \alpha ((a^2 \beta S) \gamma S) \]
\[ = b^2 \alpha ((S \beta S) \gamma a^2) \]
\[ = (S \beta S) \alpha (b^2 \gamma a^2) \]
\[ = S \alpha (b^2 \gamma a^2) \]
\[ \subseteq STP \]
\[ \subseteq P \]

that is \((a^2 \beta S) \alpha (b^2 \gamma S) \subseteq P\). It is easy to see that \(a^n \in P\), for some positive integer \(n\).

**Definition 3.9.** A \(\Gamma\)-AG-groupoid \(S\) is called \(\Gamma\)-AG-3-band if its every element satisfies
\[ a \alpha (a \beta a) = (a \alpha a) \beta a = a. \]

**Proposition 3.10.** [13] Every left identity in a \(\Gamma\)-AG-3-band is a right identity.

**Lemma 3.11.** [13] If a \(\Gamma\)-AG-3-band \(S\) has a left identity, then every left \(\Gamma\)-ideal is a \(\Gamma\)-ideal.

**Theorem 3.12.** Let \(S\) be a \(\Gamma\)-AG-3-band with left identity. Then \(P\) is a quasi-left primary ideal in \(S\) if and only if \(S\) is a quasi-primary ideal in \(S\).

**Proof.** The proof is straightforward. \(\square\)

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**REFERENCES**


