On the Classical Primary Radical Formula and Classical Primary Subsemimodules

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ABSTRACT—In this paper, we characterize the classical primary radical of subsemimodules and classical primary subsemimodules of semimodules over a commutative semirings. Furthermore we prove that if $N_j$ is a classical primary subsemimodule of $M_j$, then $N_j$ is to satisfy the classical primary radical formula in $M_j$ if and only if $M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$ is to satisfy the classical primary radical formula in $M$.

Keywords—classical primary subsemimodule, primary subsemimodule, classical primary radical, classical primary radical formula.

1. INTRODUCTION

Throughout this paper a semiring will be defined as follows: A semiring is a set $R$ together with two binary operations called addition "+" and multiplication "·" such that $(R, +)$ is a commutative semigroup and $(R, ·)$ is semigroup; connecting the two algebraic structures are the distributive laws : $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$. A semimodule $M$ over a semiring $R$ is a commutative monoid $M$ with additive identity $0$, together with a function $R \times M \rightarrow M$, defined by $(r, m) \mapsto rm$ such that:

1. $r(m + n) = rm + rn$
2. $(r + s)m = rm + sm$
3. $(rs)m = r(sm)$
4. $r0 = 0 = 0m$
5. $1m = m$

for all $m, n \in M$ and $r, s \in R$. Clearly every ring is a semiring and hence every module over a ring $R$ is a left semimodule over a semiring $R$. A nonempty subset $N$ of a $R$-semimodule $M$ is called subsemimodule of $M$ if $N$ is closed under addition and closed under scalar multiplication.

J. Saffar Ardabili S. Motmaen and A. Yousefian Darani in (2011) defined a different class of subsemimodules and called it classical prime. A proper subsemimodule $N$ of $M$ is said to be classical prime when for $a, b \in R$ and $m \in M, abm \in N$ implies that $am \in N$ or $bm \in N$. 
A proper subsemimodule $N$ of $M$ is said to be classical primary when for $a, b \in R$ and $m \in M, abm \in N$ implies that $am \in N$ or $b^m \in N$, for some positive integer $n$. A classical primary radical of $N$ in $M$, denoted by $c.prad_{M}(N)$, is defined to be the intersection of all classical primary subsemimodules containing $N$. Should there be no classical primary subsemimodule of $M$ containing $N$, then we put $c.prad_{M}(N) = M$. In this note, we shall need the notion of the envelope of a submodule introduced by R. L. McCasland and M. E. Moore in [11]. For a submodule $N$ of an $R$-module $M$, the envelope of $N$ in $M$, denoted by $E_{M}(N)$, is defined to be the subset \(\{rm : r \in R \text{ and } m \in M\}\) such that $r^k m \in N$ for some $k \in \mathbb{N}^\ast$ of $M$. Note that, in general, $E_{M}(N)$ is not an $R$-module. With the help of envelopes, the notion of the radical formula is defined as follows: a submodule $N$ of an $R$-module $M$ is said to satisfy the radical formula in $M$, if $\{E_{M}(N)\} = rad_{M}(N)$. Also, an $R$-module $M$ is said to satisfy the radical formula, if every submodule of $M$ satisfies the radical formula in $M$. The radical formula has been studied extensively by various authors (see [8], [13] and [14]).

In this paper we introduce the concept of the radical formula and study some basic properties of this class of subsemimodules. Moreover, we prove that if $N_{j}$ is a classical primary subsemimodule of $M_{j}$, then $N_{j}$ is to satisfy the classical primary radical formula in $M_{j}$ if and only if $M_{1} \times M_{2} \times \ldots \times M_{j-1} \times N_{j} \times M_{j+1} \times \ldots \times M_{n}$ is to satisfy the classical primary radical formula in $M$.

2. PRELIMINARIES

Let $R = \prod_{i=1}^{n} R_{i}$, where each $R_{i}$ is a commutative semiring with identity. Then an ideal $I = \prod_{i=1}^{n} I_{i}$ of $P$ is primary if and only if $I_{i}$ is equal to the corresponding semiring $R_{i}$ and the other is primary. Moreover, any $R$-semimodule $M$ can be uniquely decomposed into a direct product of semimodules, i.e. $M = \prod_{i=1}^{n} M_{i}$, where

$$M_{j} = (0,0,0,\ldots,0,1,0,\ldots,0)M$$

is an $R_{j}$-semimodule with action $(r_{1}, r_{2}, \ldots, r_{n})(m_{1}, m_{2}, \ldots, m_{n}) = (r_{1}m_{1}, r_{2}m_{2}, \ldots, r_{n}m_{n})$, where $r_{i} \in R_{i}$ and $m_{i} \in M_{j}$.

**Proposition 2.1.** Let $N = N_{1} \times N_{2}$ be a subsemimodule of $M$. Then $\{E_{M}(N)\} = \{E_{M_{1}}(N_{1})\} \times \{E_{M_{2}}(N_{2})\}$.

**Proof.** Let $x = \sum_{i=1}^{k}(r_{i}, s_{i})(m_{i}, n_{i}) \in \{E_{M}(N)\}$ where $(r_{i}, s_{i})^{k}(m_{i}, n_{i}) \in N_{i}$ for some $k \in \mathbb{N}^\ast$ if and only if

$$u = \sum_{i=1}^{k}r_{i}m_{i} \in \{E_{M_{1}}(N_{1})\}, \text{ with } r_{i}^{k}m_{i} \in N_{1}$$

and

$$v = \sum_{i=1}^{k}s_{i}n_{i} \in \{E_{M_{2}}(N_{2})\}, \text{ with } s_{i}^{k}n_{i} \in N_{2}.$$  

Then $x = (u, v) \in \{E_{M}(N)\}$ if and only $u \in \{E_{M_{1}}(N_{1})\}$ and $v \in \{E_{M_{2}}(N_{2})\}$ as required.
Corollary 2.2. Let $N = \prod_{i=1}^{n} N_i$ be a subsemimodule of $M$. Then $\left< E_M \left(N \right) \right> = \prod_{i=1}^{n} \left< E_{M_i} \left(N_i \right) \right>$. 

Proof. This follows from Proposition 2.1.

Proposition 2.3. If $N$ is a classical prime subsemimodule of $M$, then $\left< E_M \left(N \right) \right> = N$.

Proof. Clearly, $N \subseteq \left< E_M \left(N \right) \right>$. To show that $\left< E_M \left(N \right) \right> \subseteq N$. Let $x = \sum_{i=1}^{k} r_{i} m_{i} \in \left< E_M \left(N \right) \right>$, where $r_{i}^{k} m_{i} \in N$ for some $k \in \mathbb{N}$. Since $N$ is a classical prime subsemimodule of $M$, we have $r_{i} m_{i} \in N$. Then $x = \sum_{i=1}^{k} r_{i} m_{i} \in N$ so that $\left< E_M \left(N \right) \right> \subseteq N$. Hence $\left< E_M \left(N \right) \right> = N$.

3. CLASSICAL PRIMARY SUBSEMIMODULES

In this section, we give some characterizations for classical primary subsemimodules of $R$-semimodule $M$.

Lemma 3.1. Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. A subsemimodule $N_1 \times M_2$ is a classical primary subsemimodule of $M$ if and only if $N_1$ is a classical primary subsemimodule of $M_1$.

Proof. Suppose that $N_1 \times M_2$ is a classical primary subsemimodule of $R$-semimodule $M$. We will show that $N_1$ is a classical primary subsemimodule of $M_1$. Clearly, $N_1$ is a proper subsemimodule of $R_1$-semimodule $M_1$. To show that classical primary subsemimodule properties of $N_1$ hold, $m \in M_1$ and $a, b \in R_1$ such that $abm \in N_1$. Then $(a, 1)(b, 1)(m, n) = (abm, n) \in N_1 \times M_2$. Since $N_1 \times M_2$ is a classical primary subsemimodule of $R$-semimodule $M_1$, it follows that $(am, n) = (a, 1)(m, n) \in N_1 \times M_2$

or

$(b^n m, n) = (b, 1^n)(m, n) \in N_1 \times M_2$,

for some positive integer $n$. That is, $am \in N_1$ or $b^n m \in N_1$. Therefore $N_1$ is a classical primary subsemimodule of $R_1$-semimodule $M_1$. Conversely, suppose that $N_1$ is a classical primary subsemimodule of $R_1$-semimodule $M_1$. We will show that $N_1 \times M_2$ is a classical primary subsemimodule of $R$-semimodule $M$. Clearly, $N_1 \times M_2$ is a proper subsemimodule of $R$-semimodule $M$. To show that classical primary subsemimodule properties of $N_1 \times M_2$ hold, let $(m, n) \in M$ and $(a_1, a_2), (b_1, b_2) \in R$ such that $(a_1 b_1 m, a_2 b_2 n) = (a_1, a_2)(b_1, b_2)(m, n) \in N_1 \times M_2$. Since $N_1$ is a classical primary subsemimodule of $R_1$-semimodule $M_1$ and $a_1 b_1 m \in N_1$, we have $a_1 m \in N_1$ or $b_1^n n \in N_1$, for some positive integer $n$. Therefore $(a_1, a_2)(m, n) = (a_1 m, a_2 n) \in N_1 \times M_2$

or

$(b_1, b_2)(n, m) = (b_1^n m, b_2^n n) \in N_1 \times M_2$.

Hence $N_1 \times M_2$ is a classical primary subsemimodule of $R$-semimodule $M$.
Corollary 3.2. Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. A subsemimodule $M_1 \times N_2$ is a classical primary subsemimodule of $R$-semimodule $M$ if and only if $N_2$ is a classical primary subsemimodule of $R_2$-semimodule $M_2$.

Proof. This follows from Lemma 3.1.

Corollary 3.3. Let $M = \prod_{i=1}^{n} M_i$, where $M_i$ is an $R_i$-semimodule. A subsemimodule

$$M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$$

is a classical primary subsemimodule of $R$-semimodule $M$ if and only if $N_j$ is a classical primary subsemimodule of $R_j$-semimodule $M_j$.

Proof. This follows from Lemma 3.1 and Corollary 3.2.

Lemma 3.4. Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $N_1 \times \{n\}$ is a classical primary subsemimodule of $M_1$ then $N_1$ is a classical primary subsemimodule of $M_1$.

Proof. Suppose that $N_1 \times \{n\}$ is a classical primary subsemimodule of $R$-semimodule $M$. We will show that $N_1$ is a classical primary subsemimodule of $M_1$. Clearly, $N_1$ is a proper subsemimodule of $R_i$-semimodule $M_i$. To show that classical primary subsemimodule properties of $N_1$ hold, let $m \in M_1$ and $a, b \in R_1$ such that $abm \in N_1$. Then

$$(a,1)(b,1)(m,n) = (abm,n) \in N_1 \times \{n\}.$$  

Since $N_1 \times M_2$ is a classical primary subsemimodule of $R$-semimodule $M_1$, it follows that

$$(am,n) = (a,1)(m,n) \in N_1 \times \{n\}.$$  

or

$$(b^nm,n) = (b,1)^n(m,n) \in N_1 \times \{n\}.$$  

for some positive integer $n$. That is, $am \in N_1$ or $b^nm \in N_1$. Therefore $N_1$ is a classical primary subsemimodule of $R_i$-semimodule $M_i$.

Corollary 3.5. Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $\{n\} \times N_2$ is a classical primary subsemimodule of $R$-semimodule $M$, then $N_2$ is a classical primary subsemimodule of $R_2$-semimodule $M_2$.

Proof. This follows from Lemma 3.4.

Corollary 3.6. Let $M = \prod_{i=1}^{n} M_i$, where $M_i$ is an $R_i$-semimodule. If $\{m_1\} \times \{m_2\} \times \ldots \times N_j \times \ldots \times \{m_n\}$ is a classical primary subsemimodule of $R$-semimodule $M$, then $N_j$ is a classical primary subsemimodule of $R_j$-semimodule $M_j$.

Proof. This follows from Lemma 3.4 and Corollary 3.5.
4. RADICAL OF CLASSICAL PRIMARY SUBSEMIMODULES

A subsemimodule $N$ of an $R$-semimodule $M$ is said to satisfy the classical primary radical formula in $M$, if $\left(E_M(N)\right) = c.prad_M(N)$.

**Lemma 4.1.** Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $W$ is a classical primary subsemimodule of $R_i$-semimodule $M$ and $P = \{x \in M_1 : (x, y) \in W\}$, then $P = M_1$ or $P$ is a classical primary subsemimodule of $R_i$-semimodule $M_1$.

**Proof.** Suppose that $P \neq M_1$. We will show that $P$ is a classical primary subsemimodule of $R_i$-semimodule $M_1$. It is clear that, $P$ is a proper subsemimodule of $R_i$-semimodule $M_1$. To show that classical primary subsemimodule properties of $P$, let $a, b \in R_i$ and $m \in M_1$ such that $abm \in P$. Then $(a, 1)(b, 1)(m, y) = (abm, y) \in W$. Since $W$ is a classical primary subsemimodule of $M$, we have

$(am, l) = (a, 1)(m, y) \in W$

or

$(b^n m, y) = (b, 1)^n (m, y) \in W$,

for some positive integer $n$. It follows that $am \in P$ or $b^n m \in P$. Therefore $P$ is a classical primary subsemimodule of $M_1$.

**Corollary 4.2.** Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $W$ is a classical primary subsemimodule of $R_i$-semimodule $M$ and $P = \{x \in M_2 : (0, x) \in W\}$, then $P = M_2$ or $P$ is a classical primary subsemimodule of $R_2$-semimodule $M_2$.

**Proof.** This follows from Lemma 4.1.

**Corollary 4.3.** Let $M = \prod_{i=1}^n M_i$, where $M_i$ is an $R_i$-semimodule. If $W$ is a classical primary subsemimodule of $R_i$-semimodule $M$ and $P = \{x \in M_j : (m_1, m_2, \ldots, x, m_{j+1}, \ldots, m_n) \in W\}$, then $P = M_j$ or $P$ is a classical primary subsemimodule of $R_j$-semimodule $M_j$.

**Proof.** This follows from Lemma 4.1 and Corollary 4.2.

**Lemma 4.4.** Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule and let $N$ be a subsemimodule of $R_i$-semimodule $M_i$. Then $m \in c.prad_{M_1}(N)$ if and only if $(m, y) \in c.prad_{M_1}(N \times \{y\})$.

**Proof.** Suppose that $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. Let $N$ be a subsemimodule of $R_i$-semimodule $M_i$ and let $m \in c.prad_{M_1}(N)$.

If there is no classical primary subsemimodule of $M$ containing $N \times \{y\}$, then $c.prad_{M}(N \times \{y\}) = M$. Therefore $(m, y) \in c.prad_{M}(N \times \{y\})$.
If there is classical primary subsemimodule of $M$ containing $N \times \{y\}$, then there exists a classical primary subsemimodule $W$ with $N \times \{y\} \subseteq W$. By Lemma 4.1 and $P = \{x \in M_1 : (x, y) \in W\}$, we have $P = M_1$ or $P$ is a classical primary subsemimodule of $R_i$-semimodule $M_1$.

**Case 1:** $P = M_1$. Since $m \in c.prad_{M_1}(N)$, we have $m \in P$. Then $(m, y) \in W$. Therefore if $W$ is a classical primary subsemimodule of $M$ containing $N \times \{y\}$, then $(m, y) \in W$.

**Case 2:** $P \neq M_1$. Since $P \neq M_1$, we have $P$ is a classical primary subsemimodule of $R_i$-semimodule $M_1$. Let $x \in N$. Then $(x, y) \in N \times \{y\}$ so that $x \in P$. It follows that $N \subseteq P$. We have

$$c.rad_{M_1}(N) \subseteq c.rad_{M_1}(P) = P$$

so that $m \in P$. Therefore if $W$ is a classical primary subsemimodule of $M$ containing $N \times \{y\}$, then $(m, y) \in W$ and hence $(m, y) \in c.prad_{M_1}(N \times \{y\})$.

**Corollary 4.5.** Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule and let $N$ be a subsemimodule of $R_2$-semimodule $M_2$. Then $m \in c.prad_{M_i}(N)$ if and only if $(x, m) \in c.prad_{M_i}({x} \times N)$.

**Proof.** This follows from Lemma 4.4.

**Corollary 4.6** Let $M = \prod_{i=1}^{n} M_i$, where $M_i$ is an $R_i$-semimodule and let $N$ be a subsemimodule of $R_j$-semimodule $M_j$. Then $m \in c.prad_{M_i}(N)$ if and only if

$$(x_1, \ldots, m, x_{j+1}, \ldots, x_n) \in c.prad_{M_i}({x_1} \times {x_2} \times \ldots \times N \times {x_{j+1}} \times \ldots \times {x_n})$$

**Proof.** This follows from Lemma 4.4 and Corollary 4.5.

**Lemma 4.7.** Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $N_i$ be a subsemimodule of $R_i$-semimodule $M_i$, then $c.prad_{M_1}(N_1) \times c.prad_{M_2}(N_2) \subseteq c.prad_{M_1}(N_1 \times N_2)$.

**Proof.** Suppose that $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. Let $N_i$ be a subsemimodule of $R_i$-semimodule $M_i$. We will show that $c.prad_{M_1}(N_1) \times c.prad_{M_2}(N_2) \subseteq c.prad_{M_1}(N_1 \times N_2)$. Let

$$(x, y) \in c.prad_{M_1}(N_1) \times c.prad_{M_2}(M_2).$$

Then $x \in c.prad_{M_1}(N_1)$ and $y \in c.prad_{M_2}(N_1)$. By Lemma 4.1 and Lemma 4.4, we have

$$(x, 0) \in c.prad_{M_1}(N_1 \times \{0\}) \subseteq c.prad_{M_1}(N_1 \times N_2)$$

and

$$(0, y) \in c.prad_{M_2}({\{0\} \times N_2}) \subseteq c.prad_{M_1}(N_1 \times N_2).$$

Then $(x, y) = (x, 0) + (0, y) \in c.prad_{M_1}(N_1 \times N_2)$ and hence

$$c.prad_{M_1}(N_1) \times c.prad_{M_2}(N_2) \subseteq c.prad_{M_1}(N_1 \times N_2).$$
Corollary 4.8. Let $M = \prod_{i=1}^{n} M_i$, where $M_i$ is an $R_i$-semimodule. If $N_i$ be a subsemimodule of $R_i$-semimodule $M_i$, then
\[
\prod_{i=1}^{n} c.\text{prad}_{M_i}(N_i) \subseteq c.\text{prad}_{M}(\prod_{i=1}^{n} N_i).
\]

Proof. This follows from Lemma 4.7.

Theorem 4.9. Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $N$ is a subsemimodule of $R_i$-semimodule $M_i$, then $c.\text{prad}_{M_i}(N_i) \times c.\text{prad}_{M_i}(M_2) = c.\text{prad}_{M_i}(N_1 \times M_2)$.

Proof. Suppose that $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. Let $N$ be a subsemimodule of $R_i$-semimodule $M_i$. By Lemma 4.7, we have $c.\text{prad}_{M_i}(N_i) \times c.\text{prad}_{M_i}(M_2) \subseteq c.\text{prad}_{M}(N \times M_2)$. We will show that $c.\text{prad}_{M_i}(N_i) \times c.\text{prad}_{M_2}(M_2)$. If there is no classical primary subsemimodule of $M$ containing $N$, then $c.\text{prad}_{M_i}(N_i) = M_i$. Then
\[
c.\text{prad}_{M}(N_1 \times M_2) \subseteq c.\text{prad}_{M_i}(N_i) \times c.\text{prad}_{M_2}(M_2).
\]

If there is a classical primary subsemimodule of $M$ containing $N$, then there exists $W$ is a classical primary subsemimodule of $M_1$ containing $N$. Then $W \times M_2$ is a classical primary subsemimodule of $M_1$, containing $N \times M_2$. Let $P$ be a classical primary subsemimodule of $M$ containing $N \times M_2$. Then
\[
N \times M_2 \subseteq c.\text{prad}_{M_i}(N_i) \times M_2 = c.\text{prad}_{M_i}(N_i) \times c.\text{prad}_{M_2}(M_2).
\]

Therefore $c.\text{prad}_{M_i}(N_i) \times c.\text{prad}_{M_2}(M_2)$ and hence
\[
c.\text{prad}_{M}(N_1 \times M_2) = c.\text{prad}_{M_i}(N_i) \times c.\text{prad}_{M_2}(M_2).
\]

Corollary 4.10. Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $N$ is a subsemimodule of $R_i$-semimodule $M_2$, then $c.\text{prad}_M(M_2 \times N) = c.\text{prad}_{M_i}(M_2) \times c.\text{prad}_{M_2}(N)$.

Proof. This follows from Lemma 4.9.

Corollary 4.11. Let $M = \prod_{i=1}^{n} M_i$, where $M_i$ is an $R_i$-semimodule. If $N_i$ be a subsemimodule of $R_i$-semimodule $M_i$, then
\[
\prod_{i=1}^{n} c.\text{prad}_{M_i}(N_i) = c.\text{prad}_M(\prod_{i=1}^{n} N_i).
\]

Proof. This follows from Lemma 4.9 and Corollary 4.10.

Theorem 4.12. Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-semimodule. If $N_i$ is a classical primary subsemimodule of $M_i$, then $N_i$ is to satisfy the classical primary radical formula in $M_i$ if and only if $N_i \times M_2$ is to satisfy the classical primary radical formula in $M$.

Proof. Suppose that $N_i$ is a classical primary subsemimodule of $M_i$ and $N_i$ is to satisfy the classical primary radical formula in $M_i$. We will show that $N_i \times M_2$ is to satisfy the classical primary radical formula in $M$. Since $N_i$ is a classical primary subsemimodule of $M_i$, it follows that

\[\text{...}\]
\[ c.prad_{M_1}(N_1 \times M_2) = c.prad_{M_1}(N_1) \times c.prad_{M_1}(M_2) \]
\[ = \langle E_{M_1}(N_1) \rangle \times M_2 \]
\[ = \langle E_{M_1}(N_1 \times M_2) \rangle. \]

Therefore \( N_1 \times M_2 \) is to satisfy the classical primary radical formula in \( M \). Conversely, suppose that \( N_1 \) is a classical primary subsemimodule of \( M_1 \) and \( N_1 \times M_2 \) is to satisfy the classical primary radical formula in \( M \). We will show that \( N_1 \) is to satisfy the classical primary radical formula in \( M_1 \). Since \( N_1 \times M_2 \) is a classical primary subsemimodule of \( M \), it follows that
\[ \langle E_{M_1}(N_1) \rangle \times M_2 = \langle E_{M}(N_1 \times M_2) \rangle = c.prad_{M_1}(N_1) \times c.prad_{M_1}(M_2). \]
Then \( c.prad_{M_1}(N_1) = \langle E_{M_1}(N_1) \rangle \) and hence \( N_1 \) is to satisfy the classical primary radical formula in \( M_1 \).

**Corollary 4.13.** Let \( M = \prod_{i=1}^{n} M_i \), where \( M_i \) is an \( R_i \)-semimodule. If \( N_j \) is a classical primary subsemimodule of \( M_j \), then \( N_j \) is to satisfy the classical primary radical formula in \( M_j \) if and only if
\[ M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n \]
is to satisfy the classical primary radical formula in \( M \).

**Proof.** This follows from Theorem 4.12.

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