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2D Convection in a Plane-parallel Layer of an Ideal Gas

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ABSTRACT— ***Non-magnetic convection of an ideal compressible gas is considered in two dimensions to benchmark the code for the problems addressed which involve a rescaling of the thermodynamics from the problems originally addressed using the Pencil-code***.

Keywords— non-magnetic convection, polytropic ideal gas, compressible fluid, dynamo benchmark

1. INTRODUCTION

We study the 2D non-magnetic convection of a plane-parallel layer of a compressible fluid (an ideal gas) heated from below, following Gough [2] and Spiegel [3]. Spiegel [3] presented linear equations for the onset of convection in a plane parallel layer of perfect gas. He also gave the appropriate definition of the Rayleigh number. Gough [2] applied the results of Spiegel [3] in calculating critical Rayleigh numbers and wavenumbers for different values of the layer depth and polytropic index of static atmosphere.

2. THEORY AND RELATED WORKS

A plane-parallel layer of compressible fluid with boundary conditions imposed at the top, $z_2 = -0.1$ and bottom, $z_1 = -1.1$. At the lower and upper boundary, temperature perturbations are fixed to be zero and free-slip velocity boundary conditions are used; however, the horizontal boundary condition is periodic. We adopt Cartesian coordinates (x, z) where x denotes the horizontal direction and z is height, and gravity, $\bar{g}$, is in the direction of negative z . Our system is composed of a convection zone of depth, $d = z_2 - z_1$, embedded between two stable layers. Our study requires the implementation of more general scaling within the Pencil-Code (e.g. general choices of specific heat, c_p), compared to the scaling normally assumed (c_p = 1, Gough [2]).

The hydrostatic, thermal equilibrium solutions satisfying $\nabla p / \rho = \bar{g}$ and $\nabla^2 T = 0$, for the plane layer considered here which are the pressure, density and temperature profiles

$$p_0 = Pz^{(m+1)}, \quad \rho_0 = \frac{P}{R^* \beta_0} z^m, \quad T_0 = \beta_0 z, \tag{1}$$

where P and β_0 are integration constants. The polytropic index, $m = \frac{g_z}{R^* \beta_0} - 1$ where g_z is the acceleration due to gravity, $R^* = \frac{(\gamma - 1)}{\gamma} c_p$ is the gas constant, $\gamma = \frac{c_p}{c_v}$ is the ratio of specific heats (or adiabatic index) and z is the local depth of the plane-parallel layer. The temperature gradient, dT / dz , is given by

$$\beta_0 = \frac{\gamma}{\gamma - 1} \frac{g_0 (m+1)}{c_p}. \tag{2}$$

The initial vertical stratification is computed using polytropes of various indexes for

$$p \propto \rho^{1+1/m} \quad \text{or} \quad \rho \propto T^m . \tag{3}$$

3. COMPUTATIONAL DETAILS

In the Pencil-Code, the conservation of mass equation is implemented using the log density as

$$\frac{D \ln \rho}{Dt} = -\nabla \cdot \bar{u} \tag{4}$$

We assume that all variables are periodic in the horizontal direction and adopt the following conditions at the upper and lower boundaries:

$$\frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = 0 \quad (\text{and}\, u_z = 0). \tag{5}$$

The initial profile is specified as

$$T = \beta_0 (z - z_\infty), \tag{6}$$

$$\Phi = (z - z_\infty)(-g_z) \tag{7}$$

where Φ is a gravitational potential such that $\bar{g} = -\nabla\Phi$.

The definition of the Rayleigh number considered by Gough [2] and Spiegel [3] is

$$R_a = \frac{(g/T_0)\beta d^4}{(K/\rho_0 c_p)(\mu/\rho_0)} \tag{8}$$

where K is the thermal conductivity, μ is the shear viscosity, and β is the super adiabatic temperature gradient, $\beta = \beta_0 - g/c_p$. And the Prandtl number is a dimensionless number approximating the ratio of kinematic viscosity and thermal diffusivity. It is defined as

$$P_r = \frac{\upsilon}{\chi} = \frac{\mu c_p}{K}. \tag{9}$$

The relations between various thermodynamic and hydrodynamic parameters considered for the cases c_p = 2.5 and c_p = 1.0 are required. The general relations for an ideal gas are given by

$$c_s^2 = \gamma R^* T = \gamma p/\rho = (\gamma - 1) c_p T,$$

$$e = \frac{R^*}{(\gamma - 1)} T = \frac{c_p}{\gamma} T = c_v T,$$

$$s - s_0 = c_v \ln\left(p/\rho^\gamma\right),$$

where e is the internal energy per unit mass of fluid, s is the specific entropy, $s = c_p/c_v$ and $R^* = c_p - c_v$, where c_v is the specific heat at constant volume given by Choudhuri [1]. This adiabatic sound speed, c_s, is obtained from perturbation arguments.

4. RESULTS AND DISCUSSION

Figure 1 shows the evolution of the root mean square (rms) of the vertical velocity (u_{rms}) and maximum velocity (u_{max}) for c_p = 1.0 and 2.5 over time. The velocity can clearly be seen in images of both u_{rms} and u_{max}; the velocity increases sharply until t = 3000s and then slowly saturates. The time scale with c_p = 2.5 is faster than c_p = 1.0 by a factor of $\sqrt{20}$ = 4.47214. For c_p = 2.5, when the Rayleigh number, R_a, was increased by a factor of 1.25, the velocities grow dramatically until t $\simeq$ 400s and reaches a stable state.

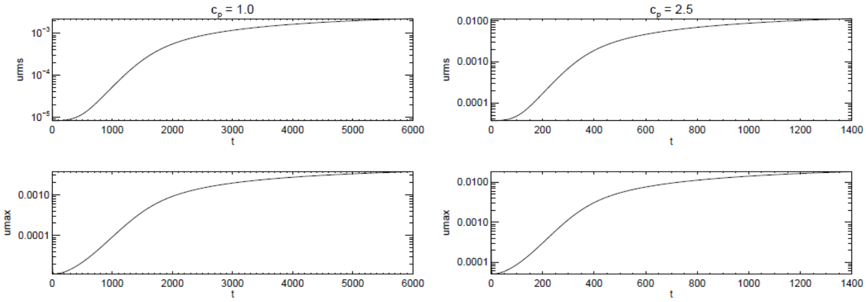

Figure 1: Plot of the root mean square of the vertical velocity (top) and the maximum velocity (bottom) with time, for c_p = 1.0 and c_p = 2.5.

Figure 2 shows vertical profiles of log density (top left), the velocity in the z-direction (top right), entropy (bottom left) and temperature (bottom right) for c_p = 1.0 and 2.5. The dashed lines represent their initial profiles. The horizontal lines are the bottom boundary at $z = -1.1$ and top boundary at $z = -0.1$.

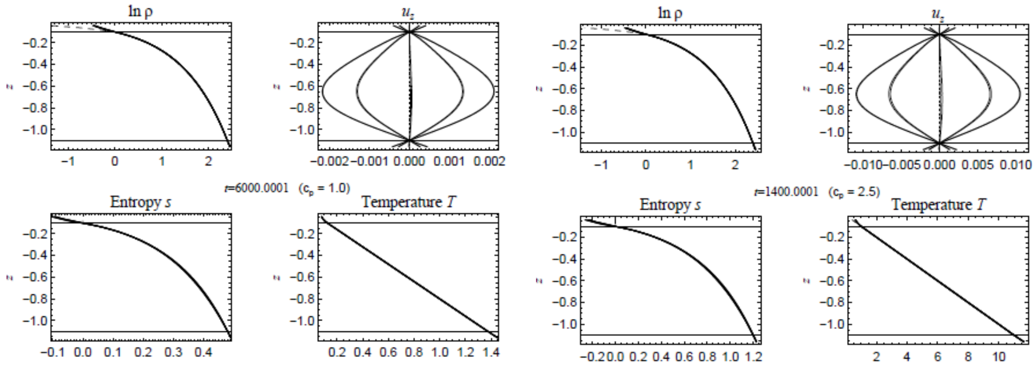

Figure2: Vertical profiles of log density (top left), vertical velocity (top right), entropy (bottom left) and temperature (bottom right) over z, for c_p = 1.0 and c_p = 2.5.

Figure 3 shows a snapshot of entropy and velocity vectors for 2D convection compared between c_p = 1.0 (t = 6000s) and c_p = 2.5 (t = 1400s) for Rayleigh number, R_a = 1189 and wave number, a_c = 2.42 (given by Gough [2]) with dark colors representing low entropy.

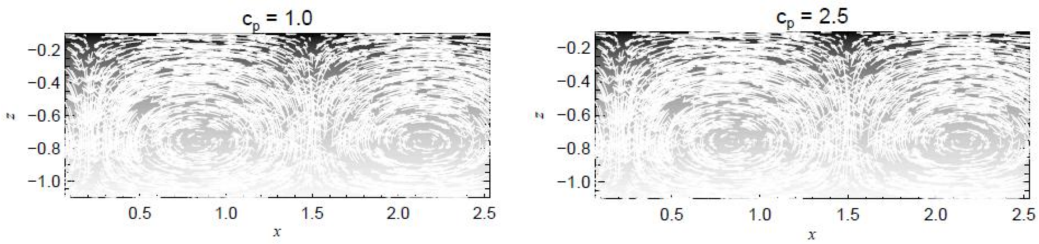

Figure3: A snapshot of velocity and entropy at time t = 6000s and t = 1400s, for comparable runs with c_p = 1 and c_p = 2.5

Table 1: Comparison of numerical values of different terms evaluated in the Pencil-Code for the two scaling considered: c_p = 2.5 and c_p = 1.0

variable	**value (c_p = 2.5)**	**value (c_p = 1.0)**	**ratio (c_p = 2.5 / c_p = 1.0)**
$\frac{du}{dt}$ (constant gravity)	-20.0	-1.0	20
$\bar{u} \cdot \nabla \bar{u}$	2.8721E-5	1.4361E-6	
$c_s^2 \left(\nabla \ln \rho + 1/c_p \nabla s \right)$	-20.0	-1.0	
c_s^2	10.3333	0.5167	
∇s	-0.80645	-0.3226	5/2
$\bar{u} \cdot \nabla s$	5.1733E-4	4.627E-5	
s	0.9123	0.3649	
T	6.2	0.775	8 [= 20/(5/2)]
$\nabla \ln \rho$	-1.6129	-1.6129	1
ρ	6.2	6.2	
R_a	1189	1189	
P_r	1	1	

5. CONCLUSION

As can be seen in Table 1 the Rayleigh number at criticality at the middle of the layer for c_p = 2.5 and c_p = 1.0, 1189, is as given in Gough [2]. Therefore, the implementation of the general thermodynamics for convection problems has been satisfied and tested. The results of Gough [2] have been reproduced, and we are in a position to extend our calculations to consider 3D geodynamo problems.

6. ACKNOWLEDGEMENT

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7. REFERENCES

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