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Dominating Sets and Domination Polynomial of Wheels

Sahib Sh. Kahat[1], Abdul Jalil M. Khalaf[2]* and Roslan Hasni[3]

[1]Department of Mathematics, Faculty of Mathematics and Computer Science
University of Kufa, Najaf, IRAQ.

[2]Department of Mathematics, Faculty of Mathematics and Computer Science
University of Kufa, Najaf, IRAQ.
E-mail: abduljaleel.khalaf {at} uokufa.edu.iq

[3]Department of Mathematics, Faculty of Science,
UMT, Terengganu, MALAYSIA.

ABSTRACT—*Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a dominating set of G, if every vertex in $V - D$ is adjacent to at least one vertex in D. Let W_n be wheel with order n. Let W_n^i be the family of dominating sets of a wheels W_n with cardinality i, and let $d(W_n, i) = |W_n^i|$. In this paper, we construct W_n, and obtain a recursive formula for $d(W_n, i)$. Using this recursive formula, we consider the polynomial $D(W_n, x) = \sum_{i=1}^{n} d(W_n, i)x^i$, which we call domination polynomial of wheels and obtain some properties of this polynomial.***

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1. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order $|V| = n$. A set $D \subseteq V$is a dominating set of G, if every vertex in $V - D$ is adjacent to atleast one vertex in D. The domination number $\gamma(G)$ is the minimumcardinality of a dominating set in G. For a detailed treatment of thisparameter, the reader is referred to [5]. It is well known and generallyaccepted that the problem of determining the dominating sets of anarbitrary graph is a difficult one (see [3]). Alikhani and Peng foundthe dominating set and domination polynomial of cycles and certaingraph [1], [2]. Kahat and Khalaf. found the dominating set and domination polynomial of stars [4]. Let G_n be graph with order n and let G_n^i bethe family of dominating sets of a graph G_n with cardinality i and let$d(G_n, i) = |G_n^i|$. We call the polynomial $D(G_n, x) = \sum_{i=\gamma(G)}^{n} d(G_n, i)x^i$, the domination polynomial of graph G [2]. Let W_n^i be the familyof dominating sets of a wheel W_n with cardinality i and let $\boldsymbol{d}(W_n, i) = |W_n^i|$. We call the polynomial $D(W_n, x) = \sum_{i=1}^{n} d(W_n, i)x^i$, thedomination polynomial of wheel.In the next section we construct the families of dominating setsof W_n with cardinality i by the families of dominating sets of W_n-1,W_n-2 and W_n-3 with cardinality $i - 1$. We investigate the dominationpolynomial of wheel in Section 3.

As usual we use $\binom{n}{i}$ for the combination n to $\boldsymbol{i}$, and we denote the set$\{\boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{n}\}$ simply by $[\boldsymbol{n}]$

2. DOMINATING SETS OF WHEEL (W_n)

Let $W_n, n \geq 3$, be the wheel with n vertices $V(W_n)$ = [n] and $E(W_n) = \{(1,2), (1,3), \dots, (1,n), (2,3), (3,4), \dots, (n-1,n), (n,2)\}$. Let W_n^i be the family of dominating sets of W_n with cardinality i. We shall investigate dominating sets of wheels. To prove our main results we need the following lemma:

Lemma 1 [4].
The following properties hold for all graph G.
(i) $|G_n^n|$= 1 (ii) $|G_n^{n-1}| = n$ (iii)$|G_n^i| = 0$ if $i > n$. (iv)$|G_n^0| = 0$

Theorem 1 [1] For every $n \geq 4, j \geq \left\lceil \frac{n}{3} \right\rceil$,$d(C_n, j) = d(C_{n-1}, j-1) + d(C_{n-2}, j-1) + d(C_{n-3}, j-1)$

Theorem 2[4]Let $S_n$be star with order $n \geq 3$, then $d(S_n, i) = d(S_{n-1}, i) + d(S_{n-1}, i-1) \forall i \neq n-2$

Theorem 3 Let $W_n$be star with order $n \geq 4$, then $d(W_n, i) = d(S_n, i) + d(C_{n-1}, i-1) \forall i \forall < n-1$

Proof. Let S_n be a star and $v \in V(S_n)$ such that v is center of S_n, let S_n be a spanning subgraph of W_n, and since $W_n - v = C_{n-1}$ then $S_n \cup C_{n-1} = W_n$, since $d(S_n, i) = |S_n^i|$, and $d(C_{n-1}, i) = |C_{n-1}^i|$, and $d(W_n, i) = |W_n^i|$., and since $d(W_n, n-1) = n$ and $d(W_n, n) = 1$ (Lemma 1), then $d(W_n, i) = d(S_n, i) + d(C_{n-1}, i)\ \forall\, i < n-1$.

Theorem 4 Let W_n be star with order $n \geq 4$, then $d(W_n, i) = d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1) + \binom{n-4}{i-1}$

Proof. By Theorem 3, and by Theorem 2[4] $d(S_n, i) = d(S_{n-1}, i) + d(S_{n-1}, i-1) = d(S_{n-2}, i) + d(S_{n-2}, i-1) + d(S_{n-1}, i-1) = d(S_{n-3}, i) + d(S_{n-3}, i-1) + d(S_{n-2}, i-1) + d(S_{n-1}, i-1)$, and by Theorem 1 [1], $d(C_n, j) = d(C_{n-1}, j-1) + d(C_{n-2}, j-1) + d(C_{n-3}, j-1)$, and since $d(S_{n-3}, i) = \binom{n-4}{i-1}$ and by Theorem 3, then

$$d(W_n, i) = d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1) + \binom{n-4}{i-1}.$$

Using Theorem 3 and Theorem 4, we obtain the coefficients of $D(W_n, x)$ for $1 \leq n \leq 15$ in Table 1. Let $d(W_n, i) = |W_n^i|$. There are interesting relationships between the numbers $d(W_n, i)$ $(1 \leq i \leq n)$ inthe table.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n															
1	1														
2	2	1													
3	3	3	1												
4	4	6	4	1											
5	1	10	10	5	1										
6	1	10	20	15	6	1									
7	1	9	29	35	21	7	1								
8	1	7	35	63	56	28	8	1							
9	1	8	36	94	118	84	36	9	1						
10	1	9	39	120	207	201	12o	45	10	1					
11	1	10	45	145	312	402	320	165	55	11	1				
12	1	11	55	176	429	693	715	484	220	66	12	1			
13	1	12	66	223	567	1074	1380	1191	703	286	78	13	1		
14	1	13	78	286	754	1565	2379	2535	1795	988	364	91	14	1	
15	1	14	91	364	1015	2212	3789	4954	4375	2863	1351	455	105	15	1

Table 2. $d(W_n, i)$ The number of dominating sets of W_n with cardinality i

In the following theorem, we obtain some properties of $d(W_n, i)$

Theorem 5 The following properties hold for every $n \in Z^+, n \geq 3$.

i. $d(W_n, 1) = 1\ \ \forall n > 4$.
ii. $d(W_n, 2) = n - 1, \forall n > 7$
iii. $d(W_n, n-2) = \binom{n}{2}$
iv. $d(W_n, n-3) = \binom{n}{3}$
v. $d(W_n, n-4) = \binom{n}{4} - (n-1)$
vi. $\gamma(W_n) = 1$.
vii. $d(W_n, i) = \binom{n}{i} - \binom{n-1}{i}\ \ \forall i < \left\lceil \frac{n-1}{3} \right\rceil$
viii. $d(W_n, i) = d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1) \forall i \geq n-2$
ix. $d(W_n, i) = d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1) + 1\ if\ i = n-3$

proof Let W_n be a wheel and $v \in V(W_n)$ such that v is center of W_n then
(i) By Theorem 3 $d(W_n, i) = d(S_n, i) + d(C_{n-1}, i-1)$, and since $d(S_n, 1) = 1\ \forall n > 4$ [4], and $d(C_{n-1}, , 1) = 0\ \forall n > 4$ [1], then $d(W_n, 1) = 1\ \forall n > 4$
(ii)Since $d(S_n, 2) = n - 1\ \forall n > 3$[4], and $d(C_{n-1}, 2) = 0\ \forall n > 7$ [1], then $d(W_n, 2) = n - 1\ \forall n > 7$
(iii) By Theorem 3, $d(W_n, n-2) = d(S_n, n-2) + d(C_{n-1}, n-2)$, and since $d(C_{n-1}, n-2) = (n-1), d(S_n, n-2) = \binom{n-1}{i-3}$[1] [4], then $d(W_n, n-2) = \binom{n-1}{n-3} - (n-1) = \frac{(n-1)(n-2)}{2} + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$
(iv) By Theorem 3, $d(W_n, n-3) = d(S_n, n-3) + d(C_{n-1}, n-3)$, and since $d(C_{n-1}, n-3) = \binom{n-1}{2}, d(S_n, n-3) = \binom{n-1}{n-4}$[1] [4], then $d(W_n, n-3) = \binom{n-1}{n-4} - \binom{n-1}{2} = \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)(n-2)}{6} = \binom{n}{3}$

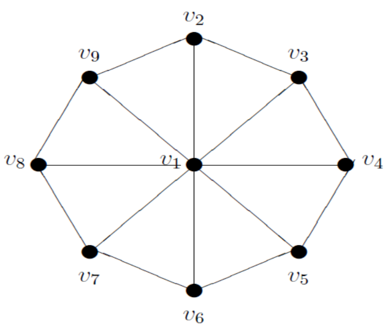
(v) By Theorem 3 and [1] [4], $d(W_n, n-4) = d(S_n, n-4) + d(C_{n-1}, n-4)$=$\binom{n}{n-4}$-$\binom{n-1}{n-4}$ + $\frac{(n-5)(n-1)n}{6}$= $\binom{n}{4}$-$\binom{n-1}{3}$ + $\frac{(n-5)(n-1)n}{6}$ =$\binom{n}{4}$-$\frac{(n-1)(n-2)(n-3)}{6}$ + $\frac{(n-5)(n-1)n}{6}$ = $\binom{n}{4} - (n-1)$
(vi) since $\{v\}$ is dominating set of $W_n \forall n \in Z^+$, then $\gamma(W_n) = 1$.
(vii)By Theorem 3, $d(W_n, i) = d(S_n, i) + d(C_{n-1}, i)$, and since$d(C_{n-1}, i) = 0 \ \forall i < \left\lceil \frac{n-1}{3} \right\rceil$ [1], then $d(W_n, i) = \binom{n}{i} - \binom{n-1}{i} \ \forall i < \left\lceil \frac{n-1}{3} \right\rceil$
(viii)By Theorem 4, $d(W_n, i) = d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1) + \binom{n-4}{i-1}$, and since $i \geq n-2 \rightarrow i-1 > n-4$, then $\binom{n-4}{i-1}$=0 therefore, $d(W_n, i) = d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1) \forall i \geq n-2$
(ix) since $i = n-3$, then $\binom{n-4}{i-1}$=$\binom{n-4}{n-4} = 1$, therefore, $d(W_n, i) = d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1)$+1 if $i = n-3$ (Theorem 4)

3. DOMINATION POLYNOMIAL OF A WHEEL

In this subsection we introduce and investigate the domination polynomial of wheels.
Definition. Let W_n^i be the family of dominating sets of a wheel$W_n$with cardinality i and let d(W_n,i)=|W_n^i|. Then the domination polynomial $D(W_n, x)$of $W_n$is defined as $D(W_n, x) = \sum_{i=1}^{n} d(W_n, i)x^i$ [1]

Theorem 6. Let $D(W_n, x)$ be domination polynomial of $W_n \forall n \geq 4$then
(i) $D(W_n, x)$= $D(S_n, x) + D(C_{n-1}, x)$-$x^{n-1}$
(ii) $D(W_n, x) = D(W_n, x) + xD(W_{n-1}, x) + xD(W_{n-2}, x) + xD(W_{n-3}, x)$+$\sum_{i=1}^{n-4}\binom{n-4}{i-1}x^i$
Proof.
(i) From definition of the domination polynomial and Theorem 3, we have
(1) $D(W_n, x) = \sum_{i=1}^{n} d(W_n, i)x^i$=$\sum_{i=1}^{n}[d(S_n, i) + d(C_{n-1}, i)]x^i$=$\sum_{i=1}^{n} d(S_n, i)x^i + \sum_{i=1}^{n} d(C_{n-1}, i)x^i$, we have $d(C_{n-1}, i)$ = 0 if $i < \left\lceil \frac{n-1}{3} \right\rceil$or $i = n$(Lemma1), then $\sum_{i=1}^{n} d(C_{n-1}, i)x^i$=$\sum_{i=\left\lceil \frac{n-1}{3} \right\rceil}^{n-1} d(C_{n-1}, i)x^i$= $D(C_{n-1}, x)$, and $\sum_{i=1}^{n} d(S_n, i)x^i$= $D(S_n, x)$, then $D(W_n, x) = D(S_n, x) + D(C_{n-1}, x)$
(2) $d(W_n, n-1)x^{n-1}$=$[d(S_n, n-1) + d(C_{n-1}, n-1)]x^{n-1}$= (n + 1)$x^{n-1} = nx^{n-1} + x^{n-1}$, but $d(W_n, n-1)x^{n-1} = nx^{n-1}$, then From (1) and (2), we get $D(W_n, x)$= $D(S_n, x) + D(C_{n-1}, x)$-$x^{n-1}$.
(ii) From definition of the domination polynomial and Theorem 3, we have
$D(W_n, x)$=$\sum_{i=1}^{n} d(W_n, i)x^i$=$\sum_{i=1}^{n}[d(W_{n-1}, i-1) + d(W_{n-2}, i-1) + d(W_{n-3}, i-1) + \binom{n-4}{i-1}]x^i$=$\sum_{i=1}^{n} d(W_{n-1}, i-1)x^i + \sum_{i=1}^{n} d(W_{n-2}, i-1)x^i + \sum_{i=1}^{n} d(W_{n-3}, i-1)x^i + \sum_{i=1}^{n}\binom{n-4}{i-1}x^i$, since $d(W_n, i) = 0$ if$i > n$or$i = 0$ by Lemma 1,
then $D(W_n, x)$=$x\sum_{i=2}^{n} d(W_{n-1}, i-1)x^{i-1}$+$x\sum_{i=2}^{n-1} d(W_{n-2}, i-1)x^{i-1}$+$x\sum_{i=2}^{n-2} d(W_{n-3}, i-1)x^{i-1}$+$\sum_{i=1}^{n-4}\binom{n-4}{i-1}x^i$= $xD(W_n, x) + xD(W_{n-1}, x) + xD(W_{n-2}, x) + xD(W_{n-3}, x)$+$\sum_{i=1}^{n-4}\binom{n-4}{i-1}x^i$

Example 1. Let W_9 be wheel with order 9, such that a vertex $(v1)$ be center vertex, $D(W_9, x) = x + 8x^2 + 36x^3 + 94x^4 + 118x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$.(see Fig-2)



$Fig-1: G = W_9 = S_9 \cup C_8$

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